SHARP DECAY ESTIMATES FOR CRITICAL DIRAC EQUATIONS

WILLIAM BORRELLI AND RUPERT L. FRANK

Abstract. We prove sharp pointwise decay estimates for critical Dirac equations on $\mathbb{R}^n$ with $n \geqslant 2$. They appear for instance in the study of critical Dirac equations on compact spin manifolds, describing blow-up profiles, and as effective equations in honeycomb structures. For the latter case, we find excited states with an explicit asymptotic behavior. Moreover, we provide some classification results both for ground states and for excited states.

1. Introduction

1.1. **Main results.** This paper is devoted to the study of solutions of the nonlinear Dirac equation

$$\mathcal{D}\,\psi = |\psi|^{2^\sharp-2}\psi \qquad \text{on} \quad \mathbb{R}^n \tag{1}$$

with the *critical exponent*

$$2^\sharp := \frac{2n}{n-1}\,,$$

as well as to certain extensions of this equation of the form

$$\mathcal{D}\,\psi = h(\psi)\,\psi \qquad \text{on} \quad \mathbb{R}^n\,, \tag{2}$$

where h is a matrix-valued function which is (approximately) homogeneous of degree $2^\# - 2$. We will always assume that $n \geqslant 2$.

As we describe below in more detail, there are at least two motivations for studying these equations, one coming from the spinorial analogue of the Yamabe problem in geometric analysis and the other one from an effective description of wave propagation in two-dimensional systems with the symmetries of a honeycomb lattice.

We are interested in two different aspects of solutions of equations (1) and (2). The first one concerns sharp bounds on the decay of solutions. The second one concerns the classification of solutions possessing some extra symmetry. The link between these two aspects is that our classification results show that our decay estimates are always sharp for 'ground state solutions' but, on the other hand, that 'excited state solutions' in general exhibit a faster decay, at least if $n = 2$.

We proceed to a precise description of our results. For $n \geqslant 2$ let $N = 2^{\left[\frac{n+1}{2}\right]}$, where $[\cdot]$ denotes the integer part of a real number, and let $\alpha_1, \ldots, \alpha_n$ be $N \times N$ Hermitian matrices

Date: January 8, 2019.

U.S. National Science Foundation grant DMS-1363432 (R.L.F.) is acknowledged.

satisfying the anticommutation relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{j,k}\,, \qquad 1 \leqslant j,k \leqslant n\,. \tag{3}$$

Such matrices exist and form a representation of the *Clifford algebra* of the Euclidean space (see e.g. [20]). Different choices of matrices satisfying (3) correspond to unitarily equivalent representations. Given a choice of matrices α_j, the Dirac operator is defined as an operator acting on functions on $\mathbb{R}^n$ with values in $\mathbb{C}^N$ by

$$\mathcal{D} := -\mathrm{i}\boldsymbol{\alpha}\cdot\nabla = -\mathrm{i}\sum_{j=1}^{n}\alpha_j\partial_{x_j}\,. \tag{4}$$

A more detailed presentation of Dirac operators and Clifford algebras can be found, for instance, in [20, 28]. We say that $\psi \in L^{2^\sharp}(\mathbb{R}^n,\mathbb{C}^N)$ is a weak solution to (1) if

$$\int_{\mathbb{R}^n}\langle\mathcal{D}\,\varphi,\psi\rangle\,dx = \int_{\mathbb{R}^n}|\psi|^{2^\sharp-2}\langle\varphi,\psi\rangle\,dx, \qquad \forall\varphi\in C_c^\infty(\mathbb{R}^n,\mathbb{C}^N). \tag{5}$$

We will give below some explicit examples of solutions of (1). Under appropriate conditions on h one can define similarly the notion of a weak solution to (2) and, modifying the arguments in [11] in a straightforward way, one can show existence of solutions also for the latter equation.

The following is the first main result of this paper.

Theorem 1.1. *Let $\psi \in L^{2^\sharp}(\mathbb{R}^n,\mathbb{C}^N)$ be a weak solution of* (1). *Then*

$$\psi\in C^\infty(\mathbb{R}^2,\mathbb{C}^2) \qquad \textit{if } n=2$$

and

$$\psi\in C^{1,\alpha}(\mathbb{R}^n,\mathbb{C}^N) \qquad \textit{for any } \alpha<2/(n-1) \textit{ if } n\geqslant 3\,.$$

Moreover, there is a constant $C<\infty$ such that

$$|\psi(x)|\leqslant\frac{C}{1+|x|^{n-1}} \qquad \textit{for all } x\in\mathbb{R}^n\,. \tag{6}$$

The decay estimate (6) is *best possible.* Indeed, in (12) and (13) below we will give explicit solutions of (1) for which $|\psi(x)|\sim\mathrm{const}\,|x|^{-n+1}$ as $|x|\to\infty$. Thus, (6) can be saturated.

The C^∞ regularity for $n=2$ was previously shown by a different argument in [39]. A weaker version of regularity for $n\geqslant 3$, namely $C^{1,\alpha}$ with some unspecified α, appears in [26].

Note also that the theorem implies that

$$\psi\in L^p(\mathbb{R}^n,\mathbb{C}^N) \qquad \text{for all } \frac{n}{n-1}<p\leqslant\infty \tag{7}$$

and

$$\psi\in L^{\frac{n}{n-1},\infty}(\mathbb{R}^n,\mathbb{C}^N)\,. \tag{8}$$

(We refer to Subsection 3.1 for the definition of weak Lebesgue spaces.) The optimality just mentioned shows that, in general,

$$\psi\notin L^{\frac{n}{n-1}}(\mathbb{R}^n,\mathbb{C}^N). \tag{9}$$

In some applications it is crucial whether solutions of (1) are square-integrable or not. Our result shows that square-integrability holds always in $n \geqslant 3$ and may not hold in $n = 2$. We will investigate the case $n = 2$ below in more detail.

We also note that if one replaces $\mathcal{D}$ by the *massive* Dirac operator then solutions exhibit exponential decay. This was shown in [9] for $n = 3$ and extended in [12] to arbitrary n; see also [15]. The fact that there are indeed solutions to the massive analogue of (1) was shown in [10].

Our proof of Theorem 1.1 consists of two steps. In a first step we prove (8) by writing (1) as an integral equation and using a bootstrap argument in Lorentz spaces. We adapt and simplify arguments by Jannelli and Solimini [27], see also [38], who studied the case of second order equations. In a second step we upgrade (8) to the pointwise bound (6). Here we prove a first order analogue of the second order result by Loiudice [30] which, in turn, is based on ideas from [37]. We refer to these papers for more references.

Remark 1.2. For the sake of simplicity, we have stated Theorem 1.1 for equation (1). *The same decay bound* (4.2) *holds, with the same proof, for solutions of* (2) provided h is an $N \times N$ matrix-valued function satisfying

$$\|h(\psi)\| \leqslant c|\psi|^{2^\# - 2} \qquad \text{for all } \psi \in \mathbb{C}^N$$

with some $c < \infty$. In fact, for equation (1) one can make use of conformal invariance and by means of a Kelvin-type transform compute the precise asymptotics of solutions. Our argument is more robust and works also for equation (2) which, in general, is not conformally invariant. This is important for applications in connection with graphene, where typically equation (2), but not (1) arises.

We now describe a class of well-known solutions of (1). For $n = 2$ we will choose

$$\alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{10}$$

For $n \geqslant 3$ we will choose the α_j of a particular block-antidiagonal form, namely, let $\sigma_1, \ldots, \sigma_n$ be $\frac{N}{2} \times \frac{N}{2}$ Hermitian matrices satisfying analogous anticommutation relations as in (3), namely,

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{j,k}, \qquad 1 \leqslant j, k \leqslant n.$$

Then the matrices

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \qquad 1 \leqslant j \leqslant n, \tag{11}$$

satisfy (3) and we shall work in the following with this choice. We write $\boldsymbol{\sigma} = (\sigma_j)_{j=1}^n$ and $a \cdot \boldsymbol{\sigma} := \sum_{j=1}^n a_j \sigma_j$ for $a \in \mathbb{R}^n$. For $n = 2$, we define $a \cdot \boldsymbol{\sigma} = a_1 + ia_2$ for $a \in \mathbb{R}^2$.

We fix a vector $\boldsymbol{n} \in \mathbb{C}^{N/2}$ with $|\boldsymbol{n}| = 1$ and a parameter $\lambda > 0$ and consider

$$\psi(x) = \lambda^{-(n-1)/2} \begin{pmatrix} V(r/\lambda)\boldsymbol{n} \\ \mathrm{i}U(r/\lambda) \left(\frac{x}{r} \cdot \boldsymbol{\sigma}\right) \boldsymbol{n} \end{pmatrix} \tag{12}$$

with

$$U(r) = n^{(n-1)/2}(1+r^2)^{-n/2} r\,, \qquad V(r) = n^{(n-1)/2}(1+r^2)^{-n/2}\,. \tag{13}$$

These functions appear, for instance, in [4]. A straightforward computation shows that they are solutions of (1). Moreover, they satisfy

$$|\psi(x)| = \lambda^{-(n-1)/2}\sqrt{V(r/\lambda)^2 + U(r/\lambda)^2} = \lambda^{-(n-1)/2} n^{(n-1)/2}(1+(r/\lambda)^2)^{-(n-1)/2}\,, \tag{14}$$

which proves the optimality statement made after Theorem 1.1.

The solutions (12), (13) are 'ground state solutions' or 'least energy solutions' of (1) in the sense that any solution $\psi \not\equiv 0$ of (1) satisfies

$$\frac{1}{2}\int_{\mathbb{R}^n} \langle \mathcal{D}\,\psi, \psi\rangle\, dx - \frac{1}{2^\sharp}\int_{\mathbb{R}^n} |\psi|^{2^\sharp}\, dx \geqslant \frac{1}{2n}\left(\frac{n}{2}\right)^n |\mathbb{S}^n|$$

with equality exactly for (12), (13). This bound was shown in [26, Proposition 4.1] and is based on inequalities by Hijazi [25] and Bär [6] after mapping equation (1) conformally to the sphere. (The equality statement made above is not explicitly stated in [26], but follows from the same arguments, taking the corresponding equality statements in Hijazi's and Bär's inequalities into account.) We also note that a simple extension of [11] to higher dimensions shows the existence of a non-trivial least energy solution. This argument does not give the above minimal value, but has the advantage of working for a more general class of nonlinearities.

Here we present a different characterization of the solutions (12), (13). Given $\boldsymbol{n} \in \mathbb{C}^{N/2}$ with $|\boldsymbol{n}| = 1$ we consider solutions of (1) of the form

$$\psi(x) = \begin{pmatrix} v(r)\boldsymbol{n} \\ \mathrm{i}u(r)\left(\frac{x}{r}\cdot\boldsymbol{\sigma}\right)\boldsymbol{n}\end{pmatrix}, \qquad r = |x|, \qquad u, v : (0,\infty) \longrightarrow \mathbb{R}\,. \tag{15}$$

This form of solutions is sometimes [13] called the *Soler/Wakano-type* ansatz. It leads to the following system

$$\begin{cases} u' + \dfrac{n-1}{r}u = v(u^2+v^2)^{1/(n-1)}\,, \\ \qquad\quad v' = -u(u^2+v^2)^{1/(n-1)}\,, \end{cases} \tag{16}$$

which needs to be supplemented by boundary conditions at the origin. To have ψ regular at the origin it is natural to require $u(0) = 0$ and then, to have a non-trivial solution, $v(0) \neq 0$ (see, e.g. [11]). We will see, however, that weaker boundary conditions suffice.

Theorem 1.3. *Let $n > 1$ and let u, v be real functions on $(0,\infty)$ satisfying (16) as well as*

$$\lim_{r\to 0} r^{(n-1)/2}u(r) = \lim_{r\to 0} r^{(n-1)/2}v(r) = 0\,.$$

Then either $u = v = 0$ or

$$u(r) = \sigma\lambda^{-(n-1)/2}U(r/\lambda)\,, \qquad v(r) = \sigma\lambda^{-(n-1)/2}V(r/\lambda)$$

for some $\lambda > 0$ and $\sigma \in \{+1,-1\}$ with U and V from (13).

Remark 1.4. The boundary conditions at the origin are necessary for the result to hold, since $u(r) = v(r) = \sqrt{(1/2)((n-1)/2)^{n-1}}\, r^{-(n-1)/2}$ is also a solution of the equation. Moreover, the same result holds, with the same proof, if the boundary condition at the origin is replaced by the condition

$$\lim_{r\to\infty} r^{(n-1)/2}u(r) = \lim_{r\to\infty} r^{(n-1)/2}v(r) = 0\,.$$

at infinity.

We next discuss 'excited state solutions' of (1) and, more generally, of (2) in the case $n = 2$. When $n = 2$, we have $N = 2$ and we write $\psi = (\psi_1, \psi_2)$. We consider the nonlinearity in (2) of the form

$$h(\psi) = \begin{pmatrix} (\beta_1|\psi_1|^2 + 2\beta_2|\psi_2|^2) & 0 \\ 0 & (\beta_1|\psi_2|^2 + 2\beta_2|\psi_1|^2) \end{pmatrix}$$

with given parameters $\beta_1, \beta_2 > 0$. As we will explain below, this particular nonlinearity arises in a problem from mathematical physics. More explicitly, we are studying the system

$$\begin{cases} (-\mathrm{i}\partial_{x_1} - \partial_{x_2})\psi_2 = (\beta_1|\psi_1|^2 + 2\beta_2|\psi_2|^2)\psi_1\,, \\ (-\mathrm{i}\partial_{x_1} + \partial_{x_2})\psi_1 = (\beta_1|\psi_2|^2 + 2\beta_2|\psi_1|^2)\psi_2\,. \end{cases} \tag{17}$$

Given $S \in \mathbb{Z}$ we look for solutions of (17) of the form

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \begin{pmatrix} v(r)e^{\mathrm{i}S\theta} \\ \mathrm{i}u(r)e^{\mathrm{i}(S+1)\theta} \end{pmatrix}, \quad x = (r\cos\theta, r\sin\theta), \quad u, v : (0,\infty) \longrightarrow \mathbb{R}\,. \tag{18}$$

Plugging this ansatz into (17) gives the system

$$\begin{cases} u' + \dfrac{S+1}{r}u = v(\beta_1 v^2 + 2\beta_2 u^2)\,, \\ \quad v' - \dfrac{S}{r}v = -u(\beta_1 u^2 + 2\beta_2 v^2)\,. \end{cases} \tag{19}$$

The following theorem shows that there is a unique (up to symmetries) solution of this system and provides precise asymptotics of u and v. In particular, we see that solutions with $S \neq 0$ have a polynomially faster decay than the ground state solution (12), (13).

Theorem 1.5. *Let $\beta_1, \beta_2 > 0$ and $S \in \mathbb{Z}$ and put*

$$a = \left(\frac{|2S+1|}{\beta_1 + 2\beta_2}\right)^{1/2} \qquad \text{and} \qquad \tau = \operatorname{sgn}(S + 1/2)\,.$$

(1) *Let $(u, v) = (Q, P)$ be the solution of (19) with*

$$Q(1) = a \qquad \text{and} \qquad P(1) = \tau a\,.$$

Then (Q, P) exists globally and satisfies for some $\ell \in (0,\infty)$, if $S + 1/2 > 0$,

$$\lim_{r\to\infty} r^{S+1}Q(r) = \lim_{r\to 0} r^{-S}P(r) = \ell$$

and

$$\lim_{r\to\infty} r^{3S+2}P(r) = \lim_{r\to 0} r^{-3S-1}Q(r) = \frac{\beta_1\ell^3}{2(2S+1)}\,,$$

and if $S+1/2<0$,

$$\lim_{r\to 0} r^{S+1}Q(r) = -\lim_{r\to\infty} r^{-S}P(r) = \ell$$

and

$$\lim_{r\to 0} r^{3S+2}P(r) = -\lim_{r\to\infty} r^{-3S-1}Q(r) = -\frac{\beta_1\ell^3}{2(2S+1)}\,,$$

Moreover,

$$\tau\, Q(r)P(r) > 0 \qquad \textit{for all } r>0$$

and

$$P(r) = (\tau/r)\, Q(1/r) \qquad \textit{for all } r>0\,.$$

(2) *If* (u,v) *is a solution of* (19) *satisfying*

$$\lim_{r\to 0} r^{1/2}u(r) = \lim_{r\to 0} r^{1/2}v(r) = 0\,,$$

then there are $\lambda>0$ *and* $\sigma\in\{-1,+1\}$ *such that*

$$u(r) = \sigma\lambda^{-1/2}Q(r/\lambda) \qquad v(r) = \sigma\lambda^{-1/2}P(r/\lambda) \qquad \textit{for all } r>0\,.$$

In the special case $\beta_1 = 2\beta_2 = 1$, that is, for (1), we will be able to write down all the solutions explicitly, see Theorem 7.1.

The methods that we develop to prove Theorems 1.3, 1.5 and 7.1 have further applications, of which we mention two. First, the methods show uniqueness up to symmetries of solutions of the form (15) for the more general class of equations (2) also in dimensions $n\geqslant 3$ under suitable assumptions on the nonlinearity h. Second, the methods allow one to classify all functions of the form (15) which satisfy (1) in $\mathbb{R}^n\setminus\{0\}$. Recall that we mentioned already one such solution in Remark 1.4. This is relevant to the spinorial analogue of the singular Yamabe problem [34, 29]. Solutions are probably of the form $r^{-(n-1)/2}$ times a periodic function of $\ln r$. This seems to be a universal feature of conformally invariant equations which, for instance, has been recently verified for a fourth order equation [19].

1.2. **First motivation: Spinorial Yamabe and Brézis–Nirenberg problems.** Equations of the form (1) appear, for instance, in the blow-up analysis of solutions of the equation

$$\mathcal{D}\,\psi = \mu\psi + |\psi|^{2^\sharp-2}\psi \qquad \text{on } M\,, \tag{20}$$

where (M,g,Σ) is a compact spin manifold, that is, a compact Riemannian manifold (M,g) carrying a spin structure Σ [28, 20]. In that case one can define a Dirac operator $\mathcal{D}$ and show that its L^2-spectrum is discrete and composed of eigenvalues of finite multiplicities accumulating at $\pm\infty$ (see, e.g., [20, 28]).

In (20), $\mu\in\mathbb{R}$ is a parameter. For $\mu=0$ the equation is referred to as the *spinorial Yamabe equation* and its study has been initiated by Ammann and collaborators [1, 4, 3, 2]; see also [24, 23, 33, 31] and references therein. Equation (20) with general $\mu\in\mathbb{R}$ is reminiscent of the Brézis–Nirenberg problem [14] and has been studied, for instance, in [26] and [8].

In particular, in [26] Isobe proved the spinorial analogue of Struwe's theorem [35] for the Brézis–Nirenberg problem. To describe this in more detail, we note that solutions of (20) are critical point of the functional

$$\mathcal{L}(\psi) = \frac{1}{2}\int_M \langle \mathcal{D}\,\psi, \psi\rangle d\,\mathrm{vol_g} - \frac{\mu}{2}\int_M |\psi|^2 d\,\mathrm{vol_g} - \frac{1}{2^\sharp}\int_{\mathbb{R}^n} |\psi|^{2^\sharp} d\,\mathrm{vol_g} \tag{21}$$

defined on $H^{\frac{1}{2}}(\Sigma M)$, the space of $H^{\frac{1}{2}}$-sections of the spinor bundle ΣM of the manifold. Here $d\,\mathrm{vol_g}$ stands for the volume measure of (M, g). Then [26, Theorem 5.2] states that any Palais–Smale sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq H^{\frac{1}{2}}(\Sigma M)$ for the functional $\mathcal{L}$ satisfies

$$\psi_n = \psi_\infty + \sum_{j=1}^{M} \omega_n^j + o(1) \qquad \text{in } H^{\frac{1}{2}}(\Sigma M), \tag{22}$$

where ψ_∞ is the weak limit of $(\psi_n)_n$ and the ω_n^j are suitably rescaled spinors obtained by mapping solutions to (1) to spinors on the manifold M. In that sense, equation (1) that we study in this paper describes bubbles in the spinorial Yamabe and Brézis–Nirenberg problems.

1.3. **Second motivation: Effective equation for graphene.** Critical Dirac equations also appear as effective models for two-dimensional physical systems related to graphene. More precisely, if $V \in C^\infty(\mathbb{R}^2, \mathbb{R})$ possesses the symmetries of a honeycomb lattice, then, as proved in [16], the dispersion bands of Schrödinger operators of the form

$$H = -\Delta + V(x) \qquad \text{in } L^2(\mathbb{R}^2)$$

exhibit generically conical intersections (the so-called *Dirac points*). This leads to the appearance of the Dirac operator as an effective operator describing, for instance, the dynamics of wave packets spectrally concentrated around such conical degeneracies. Let $u_0(x) = u_0^\varepsilon(x)$ be a wave packet spectrally concentrated around a Dirac point, that is,

$$u_0^\varepsilon(x) = \sqrt{\varepsilon}(\psi_{0,1}(\varepsilon x)\Phi_1(x) + \psi_{0,2}(\varepsilon x)\Phi_2(x)) \tag{23}$$

where Φ_j, $j = 1, 2$, are Bloch functions at a Dirac point and the functions $\psi_{0,j}$ are some (complex) amplitudes to be determined. One expects that the solution to the nonlinear Schrödinger equation with parameter $\kappa \in \mathbb{R}\setminus\{0\}$,

$$\mathrm{i}\partial_t u = -\Delta u + V(x)u + \kappa|u|^2 u\,, \tag{24}$$

with initial conditions u_0^ϵ evolves to leading order in ϵ as a modulation of Bloch functions,

$$u^\varepsilon(t,x) \underset{\epsilon\to 0^+}{\sim} \sqrt{\varepsilon}\,(\psi_1(\varepsilon t, \varepsilon x)\Phi_1(x) + \psi_2(\varepsilon t, \varepsilon x)\Phi_2(x) + \mathcal{O}(\varepsilon))\,. \tag{25}$$

As suggested by Fefferman and Weinstein in [17] the modulation coefficients ψ_j satisfy the following effective Dirac system,

$$\begin{cases} \partial_t\psi_1 + \overline{\lambda}(\partial_{x_1} + \mathrm{i}\partial_{x_2})\psi_2 = -\mathrm{i}\kappa(\beta_1|\psi_1|^2 + 2\beta_2|\psi_2|^2)\psi_1\,, \\ \partial_t\psi_2 + \lambda(\partial_{x_1} - \mathrm{i}\partial_{x_2})\psi_1 = -\mathrm{i}\kappa(\beta_1|\psi_2|^2 + 2\beta_2|\psi_1|^2)\psi_2\,, \end{cases} \tag{26}$$

with

$$\beta_1 := \int_Y |\Phi_1(x)|^4\,dx = \int_Y |\Phi_2(x)|^4\,dx\,, \qquad \beta_2 := \int_Y |\Phi_1(x)|^2|\Phi_2(x)|^2\,dx\,. \tag{27}$$

Here Y denotes a fundamental cell of the lattice and $\lambda \in \mathbb{C} \setminus \{0\}$ is a coefficient related to the potential V. The large, but finite, time-scale validity of the Dirac approximation has been proved in [18] in the linear case $\kappa = 0$ for Schwartz class intial data (23). The case of cubic nonlinearities, corresponding to (24) with $\kappa \neq 0$, is treated in [5] for high enough Sobolev regularity $H^s(\mathbb{R}^2)$ with $s > 3$.

For stationary solutions (that is, $\partial_t \psi_1 = \partial_t \psi_2 = 0$) we can write the system (26) as

$$(\alpha_1(-i\partial_{x_1}) + \alpha_2(-i\partial_{x_2})) \begin{pmatrix} \tilde{\psi_2} \\ \psi_1 \end{pmatrix} = \frac{|\kappa|}{|\lambda|} \begin{pmatrix} (\beta_1 |\tilde{\psi_2}|^2 + 2\beta_2 |\psi_1|^2)\tilde{\psi_2} \\ (\beta_1 |\psi_1|^2 + 2\beta_2 |\tilde{\psi_2}|^2)\psi_1 \end{pmatrix}$$

with α_1 and α_2 from (10) and with

$$\tilde{\psi_2} = -\frac{\kappa}{|\kappa|}\frac{\lambda}{|\lambda|}\psi_2 \,.$$

Thus, we arrive at (17) for the vector $(\tilde{\psi_2}, \psi_1)$ with coefficients $(|\kappa|/|\lambda|)\beta_j$ instead of β_j.

1.4. **Outline of the paper.** The proof of the Theorem 1.1 is achieved in several steps. First, we rewrite (1) as an integral equation, as explained in Section 2. This allows us in Section 3 to provide weak decay estimates, expressed in terms of suitable Lorentz norms. Then the desired pointwise estimates are proved in Section 4, while boundedness and regularity are the object of Section 5. Section 6 is devoted to the proof of the classification in Theorem 1.3. The remaining two sections deal with the faster decay for excited states in the two-dimensional case, first establishing an explicit family of solutions for $\beta_1 = 2\beta_2 = 1$ and then proving existence, uniqueness and asymptotics for general $\beta_1, \beta_2 > 0$.

2. An integral equation

We begin with a Liouville-type lemma.

Lemma 2.1. *Let $p \geqslant 1$ and assume that $\psi \in L^p(\mathbb{R}^n, \mathbb{C}^N)$ satisfies $\mathcal{D}\psi = 0$ in $\mathbb{R}^n$ in the sense of distributions. Then $\psi \equiv 0$.*

Proof. The corresponding result for scalar functions u satisfying $\Delta u = 0$ is well-known; see, e.g., [22]. The present result can be obtained by a simple modification of those arguments. An even simpler way is to deduce it from the scalar result. Namely, if $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$, then, since $\mathcal{D}\psi = 0$ in the sense of distributions and since $\mathcal{D}\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$,

$$-\int_{\mathbb{R}^N} \langle \Delta\varphi, \psi \rangle \, dx = \int_{\mathbb{R}^N} \langle \mathcal{D}^2\varphi, \psi \rangle \, dx = \int_{\mathbb{R}^N} \langle \mathcal{D}(\mathcal{D}\varphi), \psi \rangle \, dx = 0 \,.$$

This means that $\Delta\psi = 0$ in the sense of distributions. Applying the scalar result to each component of ψ, we obtain the assertion. □

We now use the above lemma to rewrite (1) as an integral equation. The *Green's function* Γ of the Dirac operator $\mathcal{D}$ is given by

$$\Gamma(x-y) = \frac{\mathrm{i}}{|\mathbb{S}^{n-1}|}\boldsymbol{\alpha} \cdot \frac{x-y}{|x-y|^n} \,, \tag{28}$$

where $\boldsymbol{\alpha} = (\alpha_j)_{j=1}^n$. One easily checks that this function satisfies for each fixed $y \in \mathbb{R}^n$ the equation

$$\mathcal{D}_x \, \Gamma(x-y) = \delta(x-y)\mathbb{I}_N \qquad \text{in } \mathbb{R}^n_x \tag{29}$$

in the sense of distributions.

Lemma 2.2. *If $\psi \in L^{2^\#}(\mathbb{R}^n, \mathbb{C}^N)$ solves (1) in the sense of distributions, then*

$$\psi = \Gamma * (|\psi|^{2^\sharp - 2}\psi) \,.$$

Proof. We note that $\Gamma \in L^{\frac{n}{n-1},\infty}$. Since $\psi \in L^{2^\sharp}$, we have $|\psi|^{2^\sharp-2}\psi \in L^{\frac{2n}{n+1}}$ and therefore, by the weak Young inequality (a special case of Lemma 3.2 below)

$$\tilde\psi := \Gamma * (|\psi|^{2^\sharp-2}\psi)$$

satisfies

$$\tilde\psi \in L^{2^\sharp}(\mathbb{R}^n, \mathbb{C}^N) \,.$$

Moreover, it is easy to see that

$$\mathcal{D}\tilde\psi = |\psi|^{2^\# - 2}\psi \qquad \text{in } \mathbb{R}^n$$

in the sense of distributions. This implies that

$$\mathcal{D}(\psi - \tilde\psi) = 0 \qquad \text{in } \mathbb{R}^n$$

in the sense of distributions and therefore, by Lemma 2.1, $\psi - \tilde\psi = 0$, as claimed. □

Corollary 2.3. *If $\psi \in L^{2^\#}(\mathbb{R}^n, \mathbb{C}^N)$ solves (1) in the sense of distributions, then ψ is weakly differentiable and $\nabla\psi \in L^{2n/(n+1)}(\mathbb{R}^n)$.*

We recall that ψ being weakly differentiable means that all distributional derivatives $\partial_j\psi$, $j = 1, \ldots, n$, are L^1_{loc} functions.

Proof. Since $|\psi|^{2^\#-2}\psi \in L^{2n/(n+1)}$, this follows immediately from Lemma 2.2 by the Calderon–Zygmund inequality; see, e.g., [1, Lemma 3.2.2] for the corresponding statement for Dirac operators. (Since the inequality there is stated for bounded open sets with a constant independent of the domain, it also holds for $\mathbb{R}^n$.) □

3. A weak decay estimate

Throughout this section we consider a distributional solution $\psi \in L^{2^\#}(\mathbb{R}^n, \mathbb{C}^N)$ of (1). Our goal is to show that $\psi \in L^{\frac{n}{n-1},\infty}(\mathbb{R}^n, \mathbb{C}^N)$. This should be understood as an integral version of a decay estimate. In the next section we will improve this to a pointwise decay estimate.

While the overall strategy in this section is similar to that in [27] (see also [38]) in the second order case, we believe that our arguments are more direct than in those papers.

3.1. **Reminder on Lorentz spaces.** In this subsection we collect some definitions and results about Lorentz spaces needed in the paper. We refer, for instance, to [21] and [32] for a detailed presentation.

For $\sigma \in (0,\infty), \tau \in (0,\infty]$ the *Lorentz space* $L^{\sigma,\tau}(\mathbb{R}^n) = L^{\sigma,\tau}$ is defined as the set of (equivalence classes of) measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\|f\|_{L^{\sigma,\tau}(\Omega)} < +\infty$, where

$$\|f\|_{L^{\sigma,\tau}} := \begin{cases} \left(\tau \displaystyle\int_0^\infty h^{\tau-1}\,|\{|f|>h\}|^{\tau/\sigma}\,dh\right)^{1/\tau} & \text{if} \quad \tau<\infty\,, \\ \displaystyle\sup_{h>0}\left(h\,|\{|f|>h\}|^{1/\sigma}\right) & \text{if} \quad \tau=\infty\,. \end{cases} \tag{30}$$

Here $|A|$ denotes the n-dimensional Lebesgue measure of a measurable set $A \subseteq \mathbb{R}^n$. We note that $\|f\|_{L^{\sigma,\tau}}$ is, in general, not a norm. Clearly,

$$L^{\sigma,\sigma}(\Omega) = L^\sigma(\Omega), \qquad \forall \sigma \in (0,\infty)\,, \tag{31}$$

and it is easy to see that

$$L^{\sigma,\tau_2}(\Omega) \subseteq L^{\sigma,\tau_1}(\Omega), \qquad \forall\sigma \in (0,\infty), \quad \forall \tau_1,\tau_2 \in (0,\infty], \quad \tau_1 \leqslant \tau_2. \tag{32}$$

The following results due to O'Neil [32] extend Hölder's and Young's inequalities to Lorentz spaces. We use the convention that $1/\infty = 0$.

Lemma 3.1 (Hölder inequality)**.** *Let $\sigma_1,\sigma_2,\sigma \in (0,\infty)$, $\tau_1,\tau_2,\tau \in (0,\infty]$ such that*

$$\frac{1}{\sigma_1}+\frac{1}{\sigma_2}=\frac{1}{\sigma}, \qquad \frac{1}{\tau_1}+\frac{1}{\tau_2}\geqslant\frac{1}{\tau}\,.$$

Then there is a $C>0$ such that for any $f_1 \in L^{\sigma_1,\tau_1}$ and $f_2 \in L^{\sigma_2,\tau_2}$ one has $f_1 f_2 \in L^{\sigma,\tau}$ with

$$\|f_1 f_2\|_{L^{\sigma,\tau}} \leqslant C\|f_1\|_{L^{\sigma_1,\tau_1}}\|f_2\|_{L^{\sigma_2,\tau_2}}. \tag{33}$$

Lemma 3.2 (Young inequality)**.** *Let $\sigma_1,\sigma_2,\sigma \in (1,\infty)$, $\tau_1,\tau_2,\tau \in (0,\infty]$ such that*

$$\frac{1}{\sigma_1}+\frac{1}{\sigma_2}=\frac{1}{\sigma}+1, \qquad \frac{1}{\tau_1}+\frac{1}{\tau_2}\geqslant\frac{1}{\tau}\,.$$

Then there is a $C>0$ such that for any $f_1 \in L^{\sigma_1,\tau_1}$ and $f_2 \in L^{\sigma_2,\tau_2}$ one has $f_1 * f_2 \in L^{\sigma,\tau}$ with

$$\|f_1 * f_2\|_{L^{\sigma,\tau}} \leqslant C\|f_1\|_{L^{\sigma_1,\tau_1}}\|f_2\|_{L^{\sigma_2,\tau_2}}\,. \tag{34}$$

Lemma 3.3 (A limit case of the Young inequality)**.** *Let $\sigma \in (1,\infty)$. Then there is a $C>0$ such that for any $f_1 \in L^1$ and $f_2 \in L^{\sigma,\infty}$ one has $f_1 * f_2 \in L^{\sigma,\infty}$ with*

$$\|f_1 * f_2\|_{L^{\sigma,\infty}} \leqslant C\|f_1\|_{L^1}\|f_2\|_{L^{\sigma,\infty}}\,. \tag{35}$$

In the following we will typically apply these bounds to functions with values in $\mathbb{C}^N$ or in $\mathbb{C}^{N\times N}$. These can be reduced to the case considered above by bounding $|\langle f_1(x), f_2(x)\rangle| \leqslant |f_1(x)||f_2(x)|$ and $|f_1(x)f_2(x)| \leqslant \|f_1(x)\||f_2(x)|$ when f_1 has values in $\mathbb{C}^N$ and in $\mathbb{C}^{N\times N}$, respectively, and f_2 has values in $\mathbb{C}^N$. (In the second case $\|f_1(x)\|$ denotes the operator norm of $f_1(x)$.)

3.2. **Proof of the weak decay estimate.** Again we assume that $\psi \in L^{2^\#}(\mathbb{R}^n, \mathbb{C}^N)$ is a distributional solution of (1). We first improve the a priori information $L^{2^\#}(\mathbb{R}^n, \mathbb{C}^N)$ in the Lorentz scale by lowering the second exponent.

Lemma 3.4. *For any $r > 0$,*

$$\psi \in L^{2^\#, r}(\mathbb{R}^n, \mathbb{C}^N)\,.$$

Proof. By the definition of the Lorentz norm (30), one easily finds

$$\begin{aligned} \||\psi|^{q-2}\psi\|_{q',r}^r &= r\int_0^\infty |\{|\psi|^{q-1} > \tau\}|^{r/q'}\tau^{r-1}\,d\tau = r(q-1)\int_0^\infty |\{|\psi| > \tau\}|^{r(q-1)/q}\,d\tau \\ &= \|\psi\|_{q,r(q-1)}^{r(q-1)}\,. \end{aligned} \tag{36}$$

From now on, we abbreviate $q = 2^\#$. By the integral equation for ψ from Lemma 2.2 and O'Neil's convolution inequality (34) we have

$$\|\psi\|_{q,r} = \|\Gamma * (|\psi|^{q-2}\psi)\|_{q,r} \lesssim \|\Gamma\|_{n/(n-1),\infty}\||\psi|^{q-2}\psi\|_{q',r} = \|\Gamma\|_{n/(n-1),\infty}\|\psi\|_{q,r(q-1)}^{q-1}\,.$$

In the last equality, we used (36). We apply this inequality repeatedly. We have, by assumption, $\psi \in L^q = L^{q,q}$. This and the above inequality give $\psi \in L^{q,q/(q-1)}$. The next application gives $\psi \in L^{q,q/(q-1)^2}$ and iterating this procedure, since $q/(q-1)^n \to 0$ as $n \to \infty$, we obtain the claim. $\square$

Lemma 3.5. $\psi \in L^{n/(n-1),\infty}(\mathbb{R}^n, \mathbb{C}^N)$

Proof. Again, we write $q = 2^\#$. We show that there is a constant $C > 0$ such that for any $M > 0$ we have

$$S_M := \sup\left\{\left|\int_{\mathbb{R}^n}\langle\varphi,\psi\rangle\,dx\right| :\ \|\varphi\|_{n,1} \leqslant 1\,,\ \|\varphi\|_{q'} \leqslant M\right\} \leqslant C\,. \tag{37}$$

This implies that

$$\sup\left\{\left|\int_{\mathbb{R}^n}\langle\varphi,\psi\rangle\,dx\right| :\ \|\varphi\|_{n,1} \leqslant 1\,,\ \varphi \in L^{q'}\right\} \leqslant C\,,$$

and therefore, by density and duality, $\psi \in L^{n/(n-1),\infty}$, as claimed.

We now fix $M > 0$ and aim at proving (37). Moreover, let $\varepsilon > 0$ be a parameter to be specified later. According to Lemma 3.4 we know that $\psi \in L^{q,q-2}$ and therefore, by a computation as in (36), $|\psi|^{q-2} \in L^{n,1}$. Therefore there is a bounded function f_ε, supported on a set of finite measure, such that

$$\||\psi|^{q-2} - f_\varepsilon\|_{n,1} \leqslant \varepsilon\,.$$

We denote $g_\varepsilon := |\psi|^{q-2} - f_\varepsilon$. Let $\varphi \in L^{n,1}$ with $\|\varphi\|_{n,1} \leqslant 1$ and $\|\varphi\|_{q'} \leqslant M$. We claim that

$$\int_{\mathbb{R}^n}\langle\varphi,\psi\rangle\,dx = \int_{\mathbb{R}^n}\langle\varphi,(\Gamma * (f_\varepsilon\psi))\rangle\,dx + \int_{\mathbb{R}^n}\langle\chi_\varepsilon,\psi\rangle\,dx \tag{38}$$

with

$$\chi_\varepsilon := |\psi|^{q-2}\Gamma * (g_\varepsilon(\Gamma * \varphi))\,.$$

Indeed, by the integral equation from Lemma 2.2 we have

$$\int_{\mathbb{R}^n} \langle \varphi, \psi \rangle \, dx = \int_{\mathbb{R}^n} \langle \varphi, (\Gamma * (f_\varepsilon \psi)) \rangle \, dx + \int_{\mathbb{R}^n} \langle \varphi, (\Gamma * (g_\varepsilon \psi)) \rangle \, dx \, .$$

Using Fubini's theorem and the fact that for all $x, y \in \mathbb{R}^n$ with $x \neq y$, $\Gamma(x-y)$ is an anti-Hermitian matrix one rewrites the second term on the right side as follows,

$$\begin{aligned} \int_{\mathbb{R}^n} \langle \varphi, \Gamma * (g_\varepsilon \psi) \rangle \, dx &= \int_{\mathbb{R}^n} \langle \varphi(x), \int_{\mathbb{R}^n} \Gamma(x-y)(g_\varepsilon(y)\psi(y)) \, dy \rangle \, dx \\ &= \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} dx \, \langle \varphi(x), \Gamma(x-y)(g_\varepsilon(y)\psi(y)) \rangle \\ &= -\int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} dx \, \langle \Gamma(x-y)\varphi(x), g_\varepsilon(y)\psi(y) \rangle \\ &= -\int_{\mathbb{R}^n} dy \, \langle \int_{\mathbb{R}^n} \Gamma(x-y)\varphi(x) \, dx, g_\varepsilon(y)\psi(y) \rangle \\ &= \int_{\mathbb{R}^n} \langle (\Gamma * \varphi)(y), g_\varepsilon(y)\psi(y) \rangle \, dy \, . \end{aligned}$$

Then (38) follows using again $\psi = \Gamma * (|\psi|^{q-2}\psi)$ in the last integral, and applying the same argument as above.

We now estimate the two integrals appearing on the right side of (38). By Lemmas 3.1 and 3.3 we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \langle \varphi, (\Gamma * (f_\varepsilon \psi)) \rangle \, dx \right| &\lesssim \|\varphi\|_{n,1} \|\Gamma * (f_\varepsilon \psi)\|_{n/(n-1),\infty} \leqslant \|\varphi\|_{n,1} \|\Gamma\|_{n/(n-1),\infty} \|f_\varepsilon \psi\|_1 \\ &\leqslant \|\varphi\|_{n,1} \|\Gamma\|_{n/(n-1),\infty} \|f_\varepsilon\|_{q'} \|\psi\|_q \, . \end{aligned}$$

Thus,

$$\left| \int_{\mathbb{R}^n} \langle \varphi, (\Gamma * (f_\varepsilon \psi)) \rangle \, dx \right| \leqslant C_\epsilon \, , \tag{39}$$

where C_ϵ depends, besides ϵ, only on n and $\|\psi\|_q$.

We now claim that

$$\|\chi_\varepsilon\|_{n,1} \leqslant C' \|g_\varepsilon\|_{n,1} \|\varphi\|_{n,1} \, , \qquad \|\chi_\varepsilon\|_{q'} \leqslant C' \|g_\varepsilon\|_{n,1} \|\varphi\|_{q'} \tag{40}$$

with a constant C' depending only on n and $\|\psi\|_q$. Once this is shown, we infer from the definition of S_M that

$$\left| \int_{\mathbb{R}^n} \langle \chi_\varepsilon, \psi \rangle \, dx \right| \leqslant C' \|g_\varepsilon\|_{n,1} S_M \, .$$

We now choose $\epsilon = 1/(2C')$ and recall that $\|g_\varepsilon\|_{n,1} \leqslant \epsilon$. In view of (38) and (39) we obtain

$$\left| \int_{\mathbb{R}^n} \langle \varphi, \psi \rangle \, dx \right| \leqslant C'' + \frac{1}{2} S_M \, ,$$

where C'' is C_ϵ with the above choice of ϵ. Taking the supremum over all φ, we obtain

$$S_M \leqslant C'' + \frac{1}{2} S_M \, .$$

We note that $S_M < \infty$ (in fact, since $\psi \in L^q$, we have $S_M \leqslant M \|\psi\|_q$). Therefore, the above inequality yields $S_M \leqslant 2C''$, which proves (37).

Thus, it remains to show (40). We have

$$\begin{aligned}\|\chi_\varepsilon\|_{n,1} &\leqslant \||\psi|^{q-2}\|_{n,1}\|\Gamma*(g_\varepsilon(\Gamma*\varphi))\|_\infty \lesssim \||\psi|^{q-2}\|_{n,1}\|\Gamma\|_{n/(n-1),\infty}\|g_\varepsilon(\Gamma*\varphi)\|_{n,1}\\ &\leqslant \||\psi|^{q-2}\|_{n,1}\|\Gamma\|_{n/(n-1),\infty}\|g_\varepsilon\|_{n,1}\|\Gamma*\varphi\|_\infty \lesssim \||\psi|^{q-2}\|_{n,1}\|\Gamma\|^2_{n/(n-1),\infty}\|g_\varepsilon\|_{n,1}\|\varphi\|_{n,1}.\end{aligned}$$

(Here we used $\|f*g\|_\infty \lesssim \|f\|_{n/(n-1),\infty}\|g\|_{n,1}$, which is a consequence of Hölder's inequality in Lemma 3.1 and the translation invariance of the Lorentz norms.) Since $|\psi|^{q-2}\in L^{n,1}$ by Lemma 3.4 with norm bounded by a power of $\|\psi\|_q$, we obtain the first inequality in (40).

Similarly, we have

$$\begin{aligned}\|\chi_\varepsilon\|_{q'} &\leqslant \||\psi|^{q-2}\|_{n}\|\Gamma*(g_\varepsilon(\Gamma*\varphi))\|_q \lesssim \||\psi|^{q-2}\|_{n}\|\Gamma\|_{n/(n-1),\infty}\|g_\varepsilon(\Gamma*\varphi)\|_{q'}\\ &\leqslant \||\psi|^{q-2}\|_{n}\|\Gamma\|_{n/(n-1),\infty}\|g_\varepsilon\|_{n}\|\Gamma*\varphi\|_q \lesssim \||\psi|^{q-2}\|_{n}\|\Gamma\|^2_{n/(n-1),\infty}\|g_\varepsilon\|_{n}\|\varphi\|_{q'}\,.\end{aligned}$$

Since $\|g_\varepsilon\|_n \lesssim \|g_\varepsilon\|_{n,1}$, this implies the second inequality in (40) and concludes the proof. □

4. Pointwise decay estimate

Our goal in this section is to prove the following pointwise decay estimate.

Theorem 4.1. *If $\psi\in L^{2^\#}(\mathbb{R}^n,\mathbb{C}^N)$ solves (1) in the sense of distributions, then there are constants $C,R>0$ such that*

$$|\psi(x)|\leqslant C|x|^{-(n-1)}\qquad \text{for all } |x|\geqslant R\,.$$

In fact, we will first prove a similar result about scalar functions.

Theorem 4.2. *Let $0\leqslant W\in L^{n,1}(\mathbb{R}^n)$ and let $u\in L^{n/(n-1),\infty}(\mathbb{R}^n)$ be weakly differentiable with*

$$|\nabla u|\leqslant W|u|\qquad \text{in } \mathbb{R}^n\,.$$

Then there are constants C (depending only on n) and $R>0$ (depending only on W and n) such that

$$|u(x)|\leqslant C(1+\|W\|_{n,1})\|u\|_{n/(n-1),\infty}|x|^{-n+1}\qquad \text{if } |x|\geqslant R\,.$$

In the proof of Theorem 4.2 we will make use of some ideas from [30] which, in turn, is based on ideas from [37]. Those papers deal with differential inequalities involving the Laplacian, for instance, $|\Delta u|\leqslant W|u|$ with $W\in L^{n/2,1}$, $u\in L^{n/(n-2),\infty}$, $n\geqslant 3$, and yield decay bounds $|u|\lesssim |x|^{-n+2}$. Our Theorem 4.2 is the first order analogue of these results.

The following lemma is probably well-known, but we provide a proof for the sake of completeness.

Lemma 4.3. *Let $\Omega\subset\mathbb{R}^n$ be open, let $u\in L^1_{\mathrm{loc}}(\Omega)$ be weakly differentiable and let $\rho>0$. Then almost everywhere in $\{x\in\Omega:\ \mathrm{dist}(x,\Omega^c)>\rho\}$,*

$$u(x)=\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)}u(y)\,dy+\frac{1}{|\mathbb{S}^{n-1}|}\int_{B_\rho(x)}\left(\frac{x-y}{|x-y|^n}-\frac{x-y}{\rho^n}\right)\cdot\nabla u(y)\,dy\,.$$

Proof of Lemma 4.3. We abbreviate $\Omega_\rho := \{x \in \Omega : \operatorname{dist}(x, \Omega^c) > \rho\}$. Since $u \in L^1_{\mathrm{loc}}$, the first term on the right side is a locally bounded function in Ω_ρ. Moreover, since $\nabla u \in L^1_{\mathrm{loc}}$ and since $\mathbb{1}_{B_\rho(0)}(\frac{x}{|x|^n} - \frac{x}{\rho^n}) \in L^{n/(n-1),\infty}$ has compact support, we infer from Lemma 3.3 that the second term on the right side belongs to $L^{n/(n-1),\infty}_{\mathrm{loc}}$. Therefore, by a standard mollification argument we may assume that $u \in C^1(\Omega)$. In this case we fix $x \in \Omega_\rho$ and note that for any $\omega \in \mathbb{S}^{n-1}$,

$$u(x) = u(x - \rho\omega) + \int_0^\rho \omega \cdot \nabla u(x - r\omega)\, dr\,.$$

Thus, averaging with respect to ω,

$$\begin{aligned} u(x) &= \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} u(x - \rho\omega)\, d\omega + \frac{1}{|\mathbb{S}^{n-1}|} \int_0^\rho \int_{\mathbb{S}^{n-1}} \frac{r\omega}{r^n} \cdot \nabla u(x - \rho\omega)\, d\omega\, r^{n-1}\, dr \\ &= \frac{1}{\rho^{n-1}|\mathbb{S}^{n-1}|} \int_{\partial B_\rho(x)} u(y)\, d\sigma(y) + \frac{1}{|\mathbb{S}^{n-1}|} \int_{B_\rho(0)} \frac{z}{|z|^n} \cdot \nabla u(x - z)\, dz\,. \end{aligned}$$

By the divergence theorem we have

$$\int_{B_\rho(x)} (x - y) \cdot \nabla u(y)\, dy = n \int_{B_\rho(x)} u(y)\, dy - \rho \int_{\partial B_\rho(x)} u(y)\, d\sigma(y)\,.$$

Combining the last two identities we obtain the claimed formula for $u \in C^1(\Omega)$. □

Proof of Theorem 4.2. Since both sides of the assumed inequality are homogenous with respect to u, we may assume without loss of generality that $\|u\|_{n/(n-1),\infty} = 1$. We deduce from Lemma 4.3 that

$$\begin{aligned} |u(x)| &\leqslant \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |u(y)|\, dy + \frac{1}{|\mathbb{S}^{n-1}|} \int_{B_\rho(x)} \left(\frac{1}{|x-y|^n} - \frac{1}{\rho^n} \right) |(x-y) \cdot \nabla u(y)|\, dy \\ &\leqslant \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |u(y)|\, dy + \frac{1}{|\mathbb{S}^{n-1}|} \int_{B_\rho(x)} \frac{1}{|x-y|^{n-1}} |\nabla u(y)|\, dy \\ &\leqslant \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |u(y)|\, dy + \frac{1}{|\mathbb{S}^{n-1}|} \int_{B_\rho(x)} \frac{1}{|x-y|^{n-1}} W(y)|u(y)|\, dy\,. \end{aligned}$$

In the following we will need to follow some constants and therefore we denote by A the constant (depending on n) in

$$\left| \int_{\mathbb{R}^n} f(x) g(x)\, dx \right| \leqslant A \|f\|_{n/(n-1),\infty} \|g\|_{n,1}\,.$$

(This is a special case of Lemma 3.1.) By Hölder's inequality, recalling that $\|u\|_{n/(n-1),\infty} = 1$,

$$\frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |u(y)|\, dy \leqslant A \frac{1}{|B_\rho(x)|} \|\mathbb{1}_{B_\rho(x)}\|_{n,1} = \frac{C_1}{\rho^{n-1}}\,.$$

We have thus shown that

$$|u(x)| \leqslant \frac{C_1}{\rho^{n-1}} + \frac{1}{|\mathbb{S}^{n-1}|} I_\rho(x) \tag{41}$$

with

$$I_\rho(x) = \int_{B_\rho(x)} \frac{1}{|x-y|^{n-1}} W(y)|u(y)|\, dy\,.$$

Reinserting (41) into the definition of I_r gives

$$I_\rho(x) \leqslant \frac{C_1}{\rho^{n-1}} \int_{B_\rho(x)} \frac{1}{|x-y|^{n-1}} W(y)\, dy + \frac{1}{|\mathbb{S}^{n-1}|} \int_{B_\rho(x)} \frac{1}{|x-y|^{n-1}} W(y) I_\rho(y)\, dy\,. \tag{42}$$

Again by Hölder's inequality,

$$\int_{B_\rho(x)} \frac{1}{|x-y|^{n-1}} W(y)\, dy \leqslant A \| |x|^{-n+1} \|_{n/(n-1),\infty} \| W \mathbb{1}_{B_\rho(x)} \|_{n,1} = C_2 \| W \mathbb{1}_{B_\rho(x)} \|_{n,1}\,. \tag{43}$$

In particular, the first term on the right side of (42) can be estimated by

$$\frac{C_1}{\rho^{n-1}} \int_{B_\rho(x)} \frac{1}{|x-y|^{n-1}} W(y)\, dy \leqslant \frac{C_1 C_2 \|W\|_{n,1}}{\rho^{n-1}}\,. \tag{44}$$

The second term on the right side of (42) equals

$$\begin{aligned}
&\frac{1}{|\mathbb{S}^{n-1}|} \int_{B_\rho(x)} \frac{1}{|x-y|^{n-1}} W(y) I_\rho(y)\, dy \\
&\quad = \frac{1}{|\mathbb{S}^{n-1}|} \int_{B_\rho(x)} \frac{1}{|x-y|^{n-1}} W(y) \int_{B_\rho(y)} \frac{1}{|y-z|^{n-1}} W(z) |u(z)|\, dz\, dy \\
&\quad = \frac{1}{|\mathbb{S}^{n-1}|} \int_{B_{2\rho}(x)} J_\rho(x,z) W(z) |u(z)|\, dz
\end{aligned}$$

with

$$J_\rho(x,z) = \int_{B_\rho(x)\cap B_\rho(z)} \frac{1}{|x-y|^{n-1}|y-z|^{n-1}} W(y)\, dy\,.$$

We write $B_\rho(x) \cap B_\rho(z) = A_1 \cup A_2$ with

$$A_1 = \{z \in B_\rho(x) \cap B_\rho(z) :\ |y-x| \leqslant (1/2)|x-z|\} \qquad \text{and} \qquad A_2 = (B_\rho(x) \cap B_\rho(z)) \setminus A_1\,.$$

If $z \in A_1$, then $|y-z| \geqslant |x-z| - |y-x| \geqslant (1/2)|x-z|$, and if $z \in A_2$, then $|y-x| > (1/2)|x-z|$. Thus, we can bound

$$\begin{aligned}
J_\rho(x,z) &\leqslant \frac{2^{n-1}}{|x-z|^{n-1}} \left(\int_{A_1} \frac{W(y)}{|x-y|^{n-1}}\, dy + \int_{A_2} \frac{W(y)}{|z-y|^{n-1}}\, dy \right) \\
&\leqslant \frac{2^{n-1}}{|x-z|^{n-1}} \left(\int_{B_\rho(x)} \frac{W(y)}{|x-y|^{n-1}}\, dy + \int_{B_\rho(z)} \frac{W(y)}{|z-y|^{n-1}}\, dy \right) \\
&\leqslant \frac{2^n}{|x-z|^{n-1}} \sup_{w \in B_{2\rho}(x)} \int_{B_\rho(w)} \frac{W(y)}{|w-y|^{n-1}}\, dy \\
&\leqslant \frac{2^n C_2}{|x-z|^{n-1}} \sup_{w \in B_{2\rho}(x)} \|W \mathbb{1}_{B_\rho(w)}\|_{n,1}\,,
\end{aligned}$$

where, in the last step, we used (43). To summarize, we have shown that the second term on the right side of (42) is bounded by

$$
\begin{aligned}
&\frac{1}{|\mathbb{S}^{n-1}|}\int_{B_\rho(x)}\frac{1}{|x-y|^{n-1}}W(y)I_\rho(y)\,dy\\
&\quad\leqslant \frac{2^nC_2}{|\mathbb{S}^{n-1}|}\sup_{w\in B_{2\rho}(x)}\|W\mathbb{1}_{B_\rho(w)}\|_{n,1}\int_{B_{2\rho}(x)}\frac{1}{|x-z|^{n-1}}W(z)|u(z)|\,dz\\
&\quad= \frac{2^nC_2}{|\mathbb{S}^{n-1}|}\sup_{w\in B_{2\rho}(x)}\|W\mathbb{1}_{B_\rho(w)}\|_{n,1}\left(I_\rho(x)+\int_{B_{2\rho}(x)\setminus B_\rho(x)}\frac{1}{|x-z|^{n-1}}W(z)|u(z)|\,dz\right)\\
&\quad\leqslant \frac{2^nC_2}{|\mathbb{S}^{n-1}|}\sup_{w\in B_{2\rho}(x)}\|W\mathbb{1}_{B_\rho(w)}\|_{n,1}\left(I_\rho(x)+\frac{1}{\rho^{n-1}}\int_{\mathbb{R}^n}W(z)|u(z)|\,dz\right)\\
&\quad\leqslant \frac{2^nC_2}{|\mathbb{S}^{n-1}|}\sup_{w\in B_{2\rho}(x)}\|W\mathbb{1}_{B_\rho(w)}\|_{n,1}\left(I_\rho(x)+\frac{A\|W\|_{n,1}}{\rho^{n-1}}\right).
\end{aligned}
$$

Inserting this together with (44) into (42) we obtain

$$
I_\rho(x)\leqslant \frac{C_1C_2\|W\|_{n,1}}{\rho^{n-1}}+\frac{2^nC_2}{|\mathbb{S}^{n-1}|}\sup_{w\in B_{2\rho}(x)}\|W\mathbb{1}_{B_\rho(w)}\|_{n,1}\left(I_\rho(x)+\frac{A\|W\|_{n,1}}{\rho^{n-1}}\right).
$$

Thus, for all $x\in\mathbb{R}^n$ with

$$
\frac{2^nC_2}{|\mathbb{S}^{n-1}|}\sup_{w\in B_{2\rho}(x)}\|W\mathbb{1}_{B_\rho(w)}\|_{n,1}\leqslant\frac{1}{2} \tag{45}
$$

we find

$$
I_\rho(x)\leqslant\frac{(2C_1C_2+A)\|W\|_{n,1}}{\rho^{n-1}}. \tag{46}
$$

Let us choose $R>0$ such that

$$
\frac{2^nC_2}{|\mathbb{S}^{n-1}|}\|W\mathbb{1}_{B_{R/4}(0)}\|_{n,1}\leqslant\frac{1}{2}.
$$

Then for any $x\in\mathbb{R}^n$ with $|x|\geqslant R$ the choice $\rho=|x|/4$ leads to

$$
\begin{aligned}
\frac{2^nC_2}{|\mathbb{S}^{n-1}|}\sup_{w\in B_{2\rho}(x)}\|W\mathbb{1}_{B_\rho(w)}\|_{n,1}&\leqslant\frac{2^nC_2}{|\mathbb{S}^{n-1}|}\|W\mathbb{1}_{B_{|x|-3\rho}(x)}\|_{n,1}=\frac{2^nC_2}{|\mathbb{S}^{n-1}|}\|W\mathbb{1}_{B_{|x|/4}(x)}\|_{n,1}\\
&\leqslant\frac{2^nC_2}{|\mathbb{S}^{n-1}|}\|W\mathbb{1}_{B_{R/4}(0)}\|_{n,1}\leqslant\frac{1}{2}.
\end{aligned}
$$

Thus, (45) is satisfied and therefore (46) holds with $\rho=|x|/4$. Inserting this into (41) we obtain the claimed bound. □

We finally turn to the proof of Theorem 4.1. We begin with an analogue of Lemma 4.3 for spinors.

Lemma 4.4. *Let $\Omega\subset\mathbb{R}^n$ be open, let $\psi\in L^1_{\mathrm{loc}}(\Omega,\mathbb{C}^N)$ be weakly differentiable and let $\rho>0$. Then almost everywhere in $\{x\in\Omega:\ \mathrm{dist}(x,\Omega^c)>\rho\}$,*

$$
\psi(x)=\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)}\psi(y)\,dy+\frac{\mathrm{i}}{|\mathbb{S}^{n-1}|}\int_{B_\rho(x)}\left(\frac{\boldsymbol{\alpha}\cdot(x-y)}{|x-y|^n}-\frac{\boldsymbol{\alpha}\cdot(x-y)}{\rho^n}\right)\mathcal{D}\psi(y)\,dy.
$$

Proof. By the same argument as in the proof of Lemma 4.3 it suffices to prove the formula for $\psi \in C^1(\Omega, \mathbb{C}^N)$. We fix ρ and $x \in \Omega$ with $\mathrm{dist}(x, \Omega^c) > \rho$ and choose a function $\tilde\psi \in C^1(\mathbb{R}^n)$ with compact support in Ω such that $\tilde\psi = \psi$ in $\overline{B_\rho(x)}$. (Such a function can be obtained by multiplying ψ by a cut-off function.) Using the fact that Γ from (28) is the Green's function for $\mathcal{D}$ and the Liouville theorem for spinors (Lemma 2.1), we obtain

$$\begin{aligned}\psi(x) = \tilde\psi(x) &= \int_{\mathbb{R}^n} \Gamma(x-y)\mathcal{D}\tilde\psi(y)\,dy \\ &= +\frac{\mathrm{i}}{|\mathbb{S}^{n-1}|}\int_{B_\rho(x)} \frac{\boldsymbol{\alpha}\cdot(x-y)}{|x-y|^n}\mathcal{D}\psi(y)\,dy + \frac{\mathrm{i}}{|\mathbb{S}^{n-1}|}\int_{B_\rho(x)^c} \frac{\boldsymbol{\alpha}\cdot(x-y)}{|x-y|^n}\mathcal{D}\tilde\psi(y)\,dy\,.\end{aligned}$$

We integrate by parts in the second term on the right side and obtain, using (29),

$$\begin{aligned}\frac{\mathrm{i}}{|\mathbb{S}^{n-1}|}\int_{B_\rho(x)^c} \frac{\boldsymbol{\alpha}\cdot(x-y)}{|x-y|^n}\mathcal{D}\tilde\psi(y)\,dy &= -\frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial B_\rho(x)} \frac{\boldsymbol{\alpha}\cdot(y-x)}{|y-x|}\frac{\boldsymbol{\alpha}\cdot(x-y)}{|x-y|^n}\psi(y)\,d\sigma(y) \\ &= \frac{1}{|\mathbb{S}^{n-1}|\rho^{n-1}}\int_{\partial B_\rho(x)} \psi(y)\,d\sigma(y)\,.\end{aligned}$$

On the other hand, again by integration by parts,

$$\begin{aligned}-\frac{\mathrm{i}}{|\mathbb{S}^{n-1}|}\int_{B_\rho(x)} \frac{\boldsymbol{\alpha}\cdot(x-y)}{\rho^n}\mathcal{D}\psi(y)\,dy &= -\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)}\psi(y)\,dy \\ &\quad + \frac{1}{|\mathbb{S}^{n-1}|\rho^{n-1}}\int_{\partial B_\rho(x)} \psi(y)\,d\sigma(y)\,.\end{aligned}$$

Combining these identities we obtain the claimed formula □

Proof of Theorem 4.1. Lemma 4.4 implies that

$$|\psi(x)| \leqslant \frac{1}{|B_\rho(x)|}\int_{B_\rho(x)}|\psi(y)|\,dy + \frac{1}{|\mathbb{S}^{n-1}|}\int_{B_\rho(x)}\frac{1}{|x-y|^{n-1}}|\mathcal{D}\psi(y)|\,dy\,.$$

Writing $|\mathcal{D}\psi| = Wu$ with $W = |\psi|^{2^\# - 2}$ and $u = |\psi|$ we are exactly in the same situation as in the proof of Theorem 4.2. Note that the fact that $u \in L^{n/(n-1),\infty}$ follows from Lemma 3.5 and the fact that $W \in L^{n,1}$ follows from Lemma 3.4. We omit the details of the proof. □

5. Boundedness and regularity

In order to complete the proof of Theorem 1.1 it remains to show the regularity of ψ. Note that this, together with the decay estimate from Theorem 4.1 near infinity, also yields the global upper bound (6). We also recall that $C^{1,\alpha}$ regularity with an unspecified α has already been shown in [26]. Our argument is different and uses the reformulation as an integral equation from Lemma 2.2.

As before, we assume that $\psi \in L^{2^\#}(\mathbb{R}^n, \mathbb{C}^N)$ is a distributional solution of (1).

Proposition 5.1. $\psi \in L^\infty(\mathbb{R}^n, \mathbb{C}^N)$

Proof. We denote again $q = 2^\#$.

Step 1. We show that $\psi \in L^r$ for any $q \leqslant r < \infty$ with $r^{-1} \geqslant q^{-1} - n^{-1}$. (More explicitly, $\psi \in L^r$ for any $q \leqslant r < \infty$ if $n = 2, 3$ and $\psi \in L^r$ for any $q \leqslant r \leqslant 2n/(n-3)$ if $n \geqslant 4$.)

The proof of this assertion is similar to, but simpler than that of Lemma 3.5. We only sketch the main differences. Fix r with the stated properties. It suffices to show that there is a constant $C>0$ such that for any $M>0$ we have

$$S_M := \sup\left\{\left|\int_{\mathbb{R}^n}\langle\varphi,\psi\rangle\,dx\right| :\ \|\varphi\|_{r'}\leqslant 1\,,\ \|\varphi\|_{q'}\leqslant M\right\}\leqslant C\,. \tag{47}$$

To prove this, we again decompose, for given $\varepsilon>0$, $|\psi|^{q-2}=f_\varepsilon+g_\varepsilon$ where f_ε is a bounded function supported on a set of finite measure and where $\|g_\varepsilon\|_n\leqslant\varepsilon$. We define s by $s^{-1}=r^{-1}+n^{-1}$ and note that by assumption $1<s\leqslant q$. We start again from identity (38) and estimate the first term there as follows,

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\langle\varphi,(\Gamma*(f_\varepsilon\psi))\rangle\,dx\right| &\leqslant \|\varphi\|_{r'}\|\Gamma*(f_\varepsilon\psi)\|_r \lesssim \|\varphi\|_{r'}\|\Gamma\|_{n/(n-1),\infty}\|f_\varepsilon\psi\|_s\\ &\leqslant \|\varphi\|_r\|\Gamma\|_{n/(n-1),\infty}\|f_\varepsilon\|_{qs/(q-s)}\|\psi\|_q\,.\end{aligned}$$

Thus,

$$\left|\int_{\mathbb{R}^n}\langle\varphi,(\Gamma*(f_\varepsilon\psi))\rangle\,dx\right|\leqslant C_\epsilon\,,$$

where C_ε depends, besides ε, only on n, r and $\|\psi\|_q$. To complete the proof of (47), similarly as before, it suffices to show

$$\|\chi_\varepsilon\|_{r'}\leqslant C'\|g_\varepsilon\|_n\|\varphi\|_{r'}\,,\qquad \|\chi_\varepsilon\|_{q'}\leqslant C'\|g_\varepsilon\|_n\|\varphi\|_{q'} \tag{48}$$

with a constant C' depending only on n, r and $\|\psi\|_q$. For the proof of the first inequality we bound

$$\begin{aligned}\|\chi_\varepsilon\|_{r'} &\leqslant \||\psi|^{q-2}\|_n\|\Gamma*(g_\varepsilon(\Gamma*\varphi))\|_{s'}\lesssim \||\psi|^{q-2}\|_n\|\Gamma\|_{n/(n-1),\infty}\|g_\varepsilon(\Gamma*\varphi)\|_{r'}\\ &\leqslant \||\psi|^{q-2}\|_n\|\Gamma\|_{n/(n-1),\infty}\|g_\varepsilon\|_n\|\Gamma*\varphi\|_{s'}\lesssim \||\psi|^{q-2}\|_n\|\Gamma\|^2_{n/(n-1),\infty}\|g_\varepsilon\|_n\|\varphi\|_{r'}\,.\end{aligned}$$

This proves the first inequality in (48). The second inequality in (48) was, in fact, proved while proving (40). This completes the proof of (48) and therefore that of (47).

Step 2. We show that if $\psi\in L^r$ for some $q<r<n(n+1)/(n-1)$, then $\psi\in L^s$ for $1/s=(n+1)/((n-1)r)-1/n$.

Indeed, the assumption $\psi\in L^r$ implies that $|\psi|^{q-2}\psi\in L^{(n-1)r/(n+1)}$ and therefore, by the weak Young inequality, $\Gamma*(|\psi|^{q-2}\psi)\in L^s$, where s is defined as above. (The weak Young inequality is applicable since $(n+1)/((n-1)r)-1/n>0$ by the assumed upper bound on r.) By Lemma 2.2 we obtain $\psi\in L^s$, as claimed.

Step 3. We show that if $\psi\in L^r$ for some $r>n(n+1)/(n-1)$, then $\psi\in L^\infty$.

Indeed, the assumption implies that $|\psi|^{q-2}\psi\in L^s$ for $s=(n-1)r/(n+1)>n$. On the other hand, since $\psi\in L^q$, $|\psi|^{q-2}\psi\in L^{2n/(n+1)}$ and $2n/(n+1)<n$. Thus, writing Γ as the sum of a function in $L^{s'}$ and one in $L^{2n/(n-1)}$, we obtain the assertion by Hölder's inequality.

Step 4. Let us complete the proof of the proposition.

First assume $n=2,3$. Then according to Step 1, $\psi\in L^r$ for any $r<\infty$ and therefore, by Step 3, $\psi\in L^\infty$.

Now let $n \geqslant 4$. Define $r_1^{-1} = (n-3)/(2n)$ and inductively $r_{j+1}^{-1} = (n+1)/((n-1)r_j) - 1/n$ for $j \geqslant 1$. It is elementary to check that (r_j^{-1}) is a strictly decreasing sequence which tends to $-\infty$. Thus there is a largest j, say J, such that $r_j < n(n+1)/(n-1)$. By Step 1, $\psi \in L^{r_1}$ and, by applying Step 2 repeatedly, $\psi \in L^{r_{J+1}}$. Note that $r_{J+1} \geqslant n(n+1)/(n-1)$. If this inequality is strict, we infer from Step 3 that $\psi \in L^\infty$. Finally, if $r_{J+1} = n(n+1)/(n-1)$ we apply Step 2 with $r^{-1} = r_{J+1}^{-1} + \epsilon$ where $0 < \epsilon < (n-1)^2/(n(n+1)^2)$. Then

$$\frac{1}{s} = \frac{n+1}{(n-1)r} - \frac{1}{n} = \frac{n+1}{(n-1)r_{J+1}} + \frac{(n+1)\epsilon}{n-1} - \frac{1}{n} = \frac{(n+1)\epsilon}{n-1} < \frac{n-1}{n(n+1)}$$

and therefore $\psi \in L^s$ with $s > n(n+1)/(n-1)$. By Step 3, this implies again $\psi \in L^\infty$. □

Proposition 5.2. *If $n = 2$, then $\psi \in C^\infty$. If $n \geqslant 3$, then $\psi \in C^{1,\alpha}$ for any $\alpha < 2/(n-1)$.*

Proof. By Proposition 5.1 and the assumption $\psi \in L^q$, we have $|\psi|^{q-2}\psi \in L^p$ for any $2n/(n+1) \leqslant p \leqslant \infty$. Therefore, by standard mapping properties of Riesz potentials, $\psi = \Gamma * (|\psi|^{q-2}\psi) \in C^{0,\alpha}$ for all $\alpha < 1$. This implies that $|\psi|^{q-2}\psi \in C^{0,\alpha}$ for all $\alpha < 1$ if $n = 2$ and for all $\alpha < q-2$ if $n \geqslant 3$. Therefore, by mapping properties of Riesz potentials, $\psi = \Gamma * (|\psi|^{q-2}\psi) \in C^{1,\alpha}$ for all $\alpha < 1$ if $n = 2$ and for all $\alpha < q-2$ if $n \geqslant 3$. If $n = 2$, since $q - 2 = 2$, we can iterate this argument and we eventually obtain $\psi \in C^\infty$. □

Remark 5.3. Note that the same proof shows that $\psi \in C^\infty(\mathbb{R}^n \setminus \{\psi = 0\})$. It follows from the deep results of [7] that the set $\{\psi = 0\}$ has Hausdorff dimension at most $n-2$.

6. Classification of ‘radial’ solutions

Our goal in this section is to prove Theorem 1.3 which classifies all solutions of (1) of the form (15). We emphasize that once one has passed to the radial formulation (16) the restriction that n is an integer can be dropped. Our proof works for general real $n > 1$.

Proof of Theorem 1.3. We pass to logarithmic variables and write

$$u(r) = r^{-(n-1)/2} f(\ln r)\,, \qquad v(r) = r^{-(n-1)/2} g(\ln r)$$

for functions f, g defined on $\mathbb{R}$. The equations (16) become

$$f' + \frac{n-1}{2} f = g(f^2+g^2)^{1/(n-1)}\,, \qquad g' - \frac{n-1}{2} g = -f(f^2+g^2)^{1/(n-1)}$$

and the boundary conditions become

$$\lim_{t\to-\infty} f(t) = \lim_{t\to-\infty} g(t) = 0\,.$$

We emphasize that the equations are now autonomous. Moreover, one easily checks that

$$\mathcal{E} = -fg + \frac{1}{n}(f^2+g^2)^{n/(n-1)}$$

is a constant. Since it tends to zero at $-\infty$, we conclude that

$$fg = \frac{1}{n}(f^2+g^2)^{n/(n-1)} \qquad \text{on } \mathbb{R}\,.$$

We abbreviate $\rho = f^2 + g^2$. Squaring the previous identity gives

$$f^2(\rho - f^2) = \frac{1}{n^2}\rho^{2n/(n-1)}\,.$$

Solving for f^2 we obtain

$$f^2 = \frac{1}{2}\left(\rho \pm \sqrt{\rho^2 - \frac{4}{n^2}\rho^{2n/(n-1)}}\right), \qquad g^2 = \frac{1}{2}\left(\rho \mp \sqrt{\rho^2 - \frac{4}{n^2}\rho^{2n/(n-1)}}\right).$$

Note that this also implies that $\rho^2 \geqslant (4/n^2)\rho^{2n/(n-1)}$ on $\mathbb{R}$. The signs in the formulas for f^2 and g^2 are correlated. The signs may change but they may do so only at points where $\rho^2 = (4/n^2)\rho^{2n/(n-1)}$.

Our next goal is to derive a differential equation for ρ. Using the differential equations for f and g we obtain

$$(f^2 + g^2)' = -(n-1)(f^2 - g^2)\,,$$

and inserting the above formulas for f and g we obtain

$$\rho' = \mp(n-1)\sqrt{\rho^2 - \frac{4}{n^2}\rho^{2n/(n-1)}}\,.$$

This equation can be solved explicitly. On an interval where $\rho^2 > (4/n^2)\rho^{2n/(n-1)}$ and where f^2 is given by the above formula with the $+$ sign, we find

$$\rho(t) = \left(\frac{n}{2}\right)^{n-1}\cosh^{-n+1}(t - t_0)$$

for some $t_0 \in \mathbb{R}$; see Remark 6.1 below for details. Choosing a maximal interval with theses properties we deduce that this interval has necessarily the form (t_0, ∞). Analogously one sees that any maximal interval where $\rho^2 > (4/n^2)\rho^{n/(n-1)}$ and where f^2 is given by the above formula with the $-$ sign is of the form $(-\infty, t_1)$ for some $t_1 \in \mathbb{R}$ and

$$\rho(t) = \left(\frac{n}{2}\right)^{n-1}\cosh^{-n+1}(t - t_1)$$

on this interval. We conclude that $t_0 = t_1$ and therefore

$$\rho(t) = \left(\frac{n}{2}\right)^{n-1}\cosh^{-n+1}(t - t_0) \qquad \text{for all } t \in \mathbb{R}\,,$$

unless one has $\rho^2 = (4/n^2)\rho^{2n/(n-1)}$ on all of $\mathbb{R}$, which means that either $\rho \equiv 0$ (and therefore $f \equiv g \equiv 0$) or $\rho = ((n-1)/2)^{n-1}$. In the latter case, from the formulas for f^2 and g^2 we see that $f^2 = g^2 = (1/2)((n-1)/2)^{n-1}$, but these functions do not satisfy the boundary conditions at $-\infty$, so this case is excluded. (Note that this corresponds to the singular solution in Remark 1.4.)

We return to the non-trivial case. Inserting the formula for ρ into the above formulas for f^2 and g^2 we deduce that

$$f^2 = \frac{1}{2}\left(\frac{n}{2}\right)^{n-1} e^{-(t-t_0)}\cosh^{-n}(t - t_0)\,, \qquad g^2 = \frac{1}{2}\left(\frac{n}{2}\right)^{n-1} e^{t-t_0}\cosh^{-n}(t - t_0)\,.$$

(In fact, one checks these computations separately on the intervals $(-\infty, t_0)$ and (t_0, ∞), where one knows the signs in the formulas for f^2 and g^2. The change in sign at t_0 is compensated by the fact that the expression

$$\sqrt{\rho^2 - (4/n^2)\rho^{2n/(n-1)}} = (n/2)^{n-1} \cosh^{-n}(t - t_0)|\sinh(t - t_0)|$$

involves the absolute value of the sinh.)

The formula $fg = (1/n)\rho^{n/(n-1)}$ together with the fact that ρ never vanishes implies that f and g are either both positive or both negative. Thus, for some $\sigma \in \{+1, -1\}$,

$$f = \sigma\sqrt{\frac{1}{2}\left(\frac{n}{2}\right)^{n-1}} e^{(t-t_0)/2} \cosh^{-n/2}(t - t_0)\,, \quad g = \sigma\sqrt{\frac{1}{2}\left(\frac{n}{2}\right)^{n-1}} e^{-(t-t_0)/2} \cosh^{-n/2}(t - t_0)\,.$$

Changing back to the variable r we have obtained the claimed formulas with $\lambda = e^{t_0}$. □

Remark 6.1. In the proof above we solved the equation $\rho' = \mp(n-1)\sqrt{\rho^2 - (4/n^2)\rho^{2n/(n-1)}}$. This can be done as follows, treating for instance the case with the $-$ minus. Let $\sigma(a) = \sqrt{1 - (4/n^2)a^{2/(n-1)}}$ for $0 < a < (2/n)^{-n+1}$. Then the equation reads $\rho' = -(n-1)\rho\sigma(\rho)$. We compute

$$\frac{d\sigma(a)}{da} = -\frac{1}{n-1}(1 - (4/n^2)a^{2/(n-1)})^{-1/2}(4/n^2)a^{2/(n-1)-1} = -\frac{1}{n-1}\frac{1 - \sigma(a)^2}{a\sigma(a)}\,.$$

Thus, using the equation for ρ,

$$(\sigma(\rho))' = \frac{d\sigma}{da}|_{a=\rho}\rho' = 1 - \sigma(\rho)^2\,.$$

Since $(1 - b^2)^{-1}$ has anti-derivative $\operatorname{artanh} b$, we obtain that for some t_0,

$$t - t_0 = \operatorname{artanh}\sigma(\rho(t))\,.$$

That is,

$$\tanh(t - t_0) = \sqrt{1 - (4/n^2)\rho(t)^{2/(n-1)}}\,,$$

which gives the claimed formula for $\rho(t)$.

7. Excited states in 2D

In this subsection we consider equation (1) with $n = 2$, which is the same as (17) with $\beta_1 = 1$ and $\beta_2 = 1/2$. We make the ansatz (18) and arrive at the equations

$$\begin{cases} u' + \frac{S+1}{r}u & = v(u^2 + v^2)\,, \\ v' - \frac{S}{r}v & = -u(u^2 + v^2)\,, \end{cases} \tag{49}$$

which, of course, need to be supplemented with boundary conditions. We will prove the following classification result analogous to Theorem 1.3.

Theorem 7.1. *Let $S \in \mathbb{Z}$ and let u, v be real functions on $(0, \infty)$ satisfying (49) as well as*

$$\lim_{r\to 0} r^{1/2}u(r) = \lim_{r\to 0} r^{1/2}v(r) = 0\,. \tag{50}$$

Then either $u = v = 0$ or

$$u(r) = \sigma\lambda^{-1/2}U(r/\lambda)\,, \qquad v(r) = \tau\lambda^{-1/2}V(r/\lambda)$$

for some $\lambda > 0$ *and* $\sigma, \tau \in \{+1, -1\}$*, where*

$$U(r) = \sqrt{2|2S+1|}\frac{r^S}{r^{2S+1} + r^{-(2S+1)}}, \qquad V(r) = \sqrt{2|2S+1|}\frac{r^{-S-1}}{r^{2S+1} + r^{-(2S+1)}}$$

and $\sigma = \tau$ *if* $2S+1 > 0$ *and* $\sigma = -\tau$ *if* $2S+1 < 0$.

We emphasize that as in Theorem 1.3 we impose rather weak boundary conditions.

In our proof we do not use the fact that S is an integer. Any real number S works. For $S = -1/2$ the proof shows that $u = v = 0$ is the only solution satisfying the boundary conditions.

Proof. We write again

$$u(r) = r^{-1/2} f(\ln r), \qquad v(r) = r^{-1/2} g(\ln r)$$

for functions u, v defined on $\mathbb{R}$. The equations become

$$f' + \left(S + \frac{1}{2}\right) f = g(f^2 + g^2), \qquad g' - \left(S + \frac{1}{2}\right) g = -f(f^2 + g^2)$$

and the boundary conditions become

$$\lim_{t \to -\infty} f(t) = \lim_{t \to -\infty} g(t) = 0.$$

One easily checks that

$$\mathcal{E} = -(2S+1) fg + \frac{1}{2}(f^2 + g^2)^2$$

is a constant. Since it tends to zero at $-\infty$, we conclude that

$$fg = \frac{1}{2(2S+1)}(f^2 + g^2)^2 \qquad \text{on } \mathbb{R}.$$

We abbreviate $\rho = f^2 + g^2$. Squaring the previous identity gives

$$f^2(\rho - f^2) = \frac{1}{4(2S+1)^2}\rho^4.$$

Solving for f^2 we obtain

$$f^2 = \frac{1}{2}\left(\rho \pm \sqrt{\rho^2 - \frac{1}{(2S+1)^2}\rho^4}\right), \qquad g^2 = \frac{1}{2}\left(\rho \mp \sqrt{\rho^2 - \frac{1}{(2S+1)^2}\rho^4}\right).$$

Note that this also implies that $\rho^2 \geqslant (1/(2S+1)^2)\rho^4$ on $\mathbb{R}$. The signs in the formulas for f^2 and g^2 are correlated. The signs may change but they may do so only at points where $\rho^2 = (1/(2S+1)^2)\rho^4$.

Our next goal is to derive a differential equation for ρ. Using the differential equations for f and g we obtain

$$(f^2 + g^2)' = -(2S+1)(f^2 - g^2),$$

and inserting the above formulas for f and g we obtain

$$\rho' = \mp(2S+1)\sqrt{\rho^2 - \frac{1}{(2S+1)^2}\rho^4}.$$

This equation can be solved similarly as in Remark 6.1 and we obtain, unless $\rho \equiv 0$,

$$\rho(t) = |2S+1| \cosh^{-1}((2S+1)(t-t_0)) .$$

Inserting this into the above formulas for f^2 and g^2 we deduce that

$$f^2 = \frac{|2S+1|}{2} e^{(2S+1)(t-t_0)} \cosh^{-2}((2S+1)(t-t_0))$$

and

$$g^2 = \frac{|2S+1|}{2} e^{-(2S+1)(t-t_0)} \cosh^{-2}((2S+1)(t-t_0)) .$$

(Here one has to distinguish according to whether $2S+1$ is positive or negative.) When $2S+1>0$ the formula $fg = (1/2(2S+1))\rho^2$ together with the fact that ρ never vanishes implies that f and g are either both positive or both negative. Thus, for some $\sigma \in \{+1,-1\}$,

$$f = \sigma \sqrt{\frac{|2S+1|}{2}} e^{(2S+1)(t-t_0)/2} \cosh^{-1}((2S+1)(t-t_0))$$

and

$$g = \sigma \sqrt{\frac{|2S+1|}{2}} e^{-(2S+1)(t-t_0)/2} \cosh^{-1}((2S+1)(t-t_0)) .$$

Changing back to the variable r we have obtained the claimed formulas with $\lambda = e^{t_0}$. In case $2S+1<0$ is similar, but f and g have opposite signs. □

8. Faster decay for excited states

The aim of this section is to prove Theorem 1.5 concerning (17) with parameters $\beta_1, \beta_2 > 0$. The case $\beta_1 = 2\beta_2 = 1$ was treated in the previous section and the case $\beta_1 = 2\beta_2$ can be reduced to the former by scaling.

When $\beta_1 \neq 2\beta_2$ we cannot provide explicit solutions, but we can still prove existence and uniqueness (up to symmetries) of a solution and can study its asymptotic behavior rather precisely.

Proof of Theorem 1.5. Step 1. Introducing logarithmic variables. We set again

$$u(r) = r^{-1/2} f(\ln r) , \qquad v(r) = r^{-1/2} g(\ln r) \tag{51}$$

for functions f, g defined on $\mathbb{R}$, so that system (19) becomes

$$\begin{cases} f' + \left(S+\frac{1}{2}\right) f &= g(2\beta_2 f^2 + \beta_1 g^2) , \\ g' - \left(S+\frac{1}{2}\right) g &= -f(\beta_1 f^2 + 2\beta_2 g^2) \end{cases} \tag{52}$$

and the boundary conditions in the theorem read

$$\lim_{t\to-\infty} f(t) = \lim_{t\to-\infty} g(t) = 0 . \tag{53}$$

One easily checks that

$$\mathcal{E} = \frac{\beta_1}{4}(f^4+g^4) + \beta_2 f^2 g^2 - \left(S+\frac{1}{2}\right) fg \tag{54}$$

is constant. The boundary conditions (53) imply that $\mathcal{E} = 0$, that is,

$$\frac{\beta_1}{4}(f^4 + g^4) + \beta_2 f^2 g^2 = \left(S + \frac{1}{2}\right) fg. \tag{55}$$

This implies, in particular, that $f(t) \neq 0$ for all $t \in \mathbb{R}$ and $g(t) \neq 0$ for all $t \in \mathbb{R}$, unless $f \equiv g \equiv 0$. (Indeed, if $f(t_0) = 0$, then (55) implies $g(t_0) = 0$ and then (52) implies $f \equiv g \equiv 0$. The argument for g is similar.) Moreover, it implies that

$$\tau\, f(t)g(t) > 0 \qquad \text{for all } t \in \mathbb{R}\,.$$

Step 2. Monotonicity of the angle. We shall show that, if $(f, g) \neq (0, 0)$ is a solution of (52) with $\mathcal{E} = 0$, then (f, g) is global and the limits

$$\theta_\pm := \lim_{t\to\pm\infty} \arctan \frac{g(t)}{f(t)}$$

exist and satisfy $\theta_+ < \theta_-$.

Indeed, the fact that $\mathcal{E} = 0$ on the maximal interval of existence easily implies that the solution is global. Moreover, as remarked in the previous step, $\mathcal{E} = 0$ implies that f and g never vanish and therefore we can introduce

$$\theta(t) = \arctan \frac{g(t)}{f(t)}\,.$$

Using (52) and (55) we compute

$$\theta' = \frac{g'f - f'g}{f^2 + g^2} = \frac{(2S+1)gf - \beta_1(f^4 + g^4) - 4\beta_2 f^2 g^2}{f^2 + g^2} = \frac{-(\beta_1/2)(f^4 + g^4) - 2\beta_2 f^2 g^2}{f^2 + g^2} < 0\,.$$

This proves the claim.

Step 3. Asymptotics of solutions. We shall show that any solution (f, g) of (52) with $\mathcal{E} = 0$ is global and satisfies (53) and

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0\,. \tag{56}$$

The global existence was already shown in the previous step. We shall deduce the asymptotic behavior from the Poincaré–Bendixson theorem in the form given, for instance, in [36, Theorem 7.16]. Let

$$\omega_\pm = \{(x, y) \in \mathbb{R}^2 : \text{ for some } t_n \to \pm\infty\,, (f(t_n), g(t_n)) \to (x, y)\}\,.$$

Since the set $\{(x, y) : (\beta_1/4)(x^4 + y^4) + \beta_2 x^2 y^2 = (S + 1/2)xy\}$ is compact, it is easy to see that the sets $\omega_\pm$ are non-empty, compact and connected [36, Lemma 6.6]. According to Poincaré–Bendixson, for each one of the signs $\pm$, one of the following three alternatives holds: (a) $\omega_\pm$ is a fixed point, (b) $\omega_\pm$ is a regular periodic orbit, (c) $\omega_\pm$ consists of fixed points and non-closed orbits connecting these fixed points.

A simple computation shows that the only constant solution of (52) with $\mathcal{E} = 0$ is $(f, g) \equiv (0, 0)$. Thus, if alternative (a) holds for both signs $\pm$, then we are done. Let us rule out (b) and (c). Note that in both cases (b) and (c), the limiting periodic orbit and the limiting homoclinic orbits, if they would exist, would have $\mathcal{E} = 0$.

According to Step 2, there are no non-trivial periodic solutions of (52) with $\mathcal{E} = 0$ (because for a non-trivial periodic solution $(\tilde{f}, \tilde{g})$, $\arctan(\tilde{g}/\tilde{f})$ does not have a limit). This rules out (b).

According to Step 2, there are $\theta_\pm$ such that

$$\omega_\pm \subset \{(r\cos\theta_\pm, r\sin\theta_\pm) : \ r \geqslant 0\} .$$

Thus, in order to rule out (c), it suffices to rule out the existence of a non-trivial solution $(\tilde{f}, \tilde{g})$ of (52) with $(\tilde{f}(t), \tilde{g}(t)) \to (0,0)$ for $|t| \to \infty$ and such that $\arctan(\tilde{g}(t)/\tilde{f}(t)) = \theta_\pm$ for all t. But this is again ruled out by Step 2. This completes the proof of the assertion.

Step 4. Existence of a homoclinic orbit. Let a and τ be as in the theorem and consider the solution $(f, g) = (q, p)$ of (52) with initial values

$$q(0) = \tau\, p(0) = a .$$

We shall show that this solution is global and satisfies the asymptotic conditions (53) and (56).

Indeed, by definition of a, identity (55) is satisfied. Therefore, the solution has $\mathcal{E} = 0$. The rest now follows from Step 3.

Step 5. Exponential decay. We shall show that for any solution (f, g) of (52) satisfying (53) there is a constant C such that

$$(f^2 + g^2)^{1/2} \leqslant Ce^{-|S+1/2||t|} \qquad \text{for all } t \in \mathbb{R} .$$

Indeed, we compute, using (52),

$$(f^2 + g^2)' = 2(ff' + gg') = (-(2S+1) + (2\beta_2 - \beta_1)fg)(f^2 - g^2) ,$$

$$(f^2 - g^2)' = 2(ff' - gg') = (-(2S+1) + 2(2\beta_2 + \beta_1)fg)(f^2 + g^2)$$

and

$$(fg)' = f'g + fg' = -\beta_1(f^4 - g^4) .$$

This implies that

$$\begin{aligned}(f^2 + g^2)'' = {} & (-(2S+1) + (2\beta_2 - \beta_1)fg)(-(2S+1) + 2(2\beta_2 + \beta_1)fg)(f^2 + g^2) \\ & - (2\beta_2 - \beta_1)\beta_1(f^4 - g^4)(f^2 - g^2) .\end{aligned}$$

We set $\psi = f^2 + g^2$ and write the previous equation as

$$-\psi'' + V\psi = -(2S+1)^2\psi$$

with

$$V = -(2S+1)(6\beta_2 + \beta_1)fg + (2\beta_2 - \beta_1)2(2\beta_2 + \beta_1)f^2g^2 - (2\beta_2 - \beta_1)\beta_1(f^2 - g^2)^2 .$$

By (53) we have $\mathcal{E} = 0$ and therefore, by Step 3, $V(t) \to 0$ as $|t| \to \infty$. By a standard comparison argument this implies that for any $0 < \epsilon \leqslant (2S+1)^2$ there is a C_ϵ such that

$$\psi(t) \leqslant C_\epsilon e^{-\sqrt{(2S+1)^2 - \epsilon}|t|} \qquad \text{for all } t \in \mathbb{R} . \tag{57}$$

For the sake of completeness we provide the details of this argument. Given $0<\epsilon\leqslant(2S+1)^2$ we choose $T_\epsilon<\infty$ such that $V(t)\geqslant-\epsilon$ for $t\geqslant T_\epsilon$. The function

$$\varphi(t)=\psi(t)-\psi(T_\epsilon)e^{-\sqrt{(2S+1)^2-\epsilon}(t-T_\epsilon)}$$

satisfies $\varphi(T_\epsilon)=0$, $\lim_{t\to\infty}\varphi(t)=0$ and

$$\varphi''\geqslant((2S+1)^2-\epsilon)\varphi \qquad \text{in } (T_\epsilon,\infty)\,.$$

By the maximum principle, this implies that $\varphi\leqslant 0$ in $[T_\epsilon,\infty)$. Similarly, one proves a bound near $-\infty$ and the remaining bound is obtained by continuity. This proves (57).

Because of the decay (57) we can apply the Green's function to the equation for ψ and obtain

$$\psi(t)=-\frac{1}{2|2S+1|}\int_{\mathbb{R}}e^{-|2S+1||t-t'|}V(t')\psi(t')\,dt'\,.$$

Using this equation and the apriori bound (57) it is easy to obtain the claimed bound for ψ.

Step 6. Asymptotic behavior of f and g. Again, we let $(f,g)\not\equiv(0,0)$ be a solution of (52) satisfying (53). We shall show that

$$\ell:=\lim_{t\to\tau\infty}e^{(S+1/2)t}f(t) \text{ and } \ell':=\lim_{t\to-\tau\infty}e^{-(S+1/2)t}g(t) \quad \text{exist and are non-zero and finite}$$

and that

$$\lim_{t\to\tau\infty}e^{3(S+1/2)t}g(t)=\frac{\beta_1\ell^3}{4(S+1/2)} \quad\text{and}\quad \lim_{t\to-\tau\infty}e^{-3(S+1/2)t}f(t)=\frac{\beta_1\ell'^3}{4(S+1/2)}\,.$$

Let us prove this in case $S+1/2>0$ (so $\tau=+1$), the case $S+1/2<0$ being similar. The function $F(t)=e^{(S+1/2)t}f(t)$ satisfies

$$F'(t)=e^{(S+1/2)t}(f'+(S+1/2)f)=e^{(S+1/2)t}g(2\beta_2f^2+\beta_1g^2)\,.$$

As shown in Step 1, either f and g are both positive or both negative. Thus, either F is positive and increasing or it is negative and decreasing. Since it is bounded by Step 5, it tends in any case to a finite, non-zero limit ℓ. This proves the first assertion.

The function $G(t)=e^{-(S+1/2)t}g(t)$ satisfies

$$G'(t)=e^{-(S+1/2)t}(g'-(S+1/2)g)=-e^{-(S+1/2)t}f(\beta_1f^2+2\beta_2g^2)\,. \tag{58}$$

For the sake of simplicity we now assume that f and g are both positive. The case where both are negative is treated similarly. Given $0<\epsilon\leqslant\ell$ there is a $t_\epsilon<\infty$ such that $f(t)\geqslant(\ell-\epsilon)e^{-(S+1/2)t}$ for $t\geqslant t_\epsilon$. We bound the right side of (58) and get

$$G'(t)\leqslant-\beta_1(\ell-\epsilon)^3e^{-4(S+1/2)t} \qquad \text{for all } t\geqslant t_\epsilon\,,$$

and, since $G(t)\to0$ as $t\to\infty$ by Step 5,

$$G(t)=-\int_t^\infty G'(s)\,ds\geqslant\beta_1(\ell-\epsilon)^3\int_t^\infty e^{-4(S+1/2)s}\,ds=\frac{\beta_1(\ell-\epsilon)^3}{4(S+1/2)}e^{-4(S+1/2)t} \text{ for all } t\geqslant t_\epsilon$$

and

$$g(t)\geqslant\frac{\beta_1(\ell-\epsilon)^3}{4(S+1/2)}e^{-3(S+1/2)t} \qquad \text{for all } t\geqslant t_\epsilon\,.$$

This is the desired asymptotic lower bound on $g(t)$. The proof of the upper bound is similar, but slightly more complicated. Using the bounds from Step 5 in (58), we get

$$G'(t) \geqslant -\operatorname{const} e^{-4(S+1/2)t} \qquad \text{for all } t \in \mathbb{R}$$

and therefore, by a similar argument as before,

$$g(t) \leqslant \operatorname{const} e^{-3(S+1/2)t} \qquad \text{for all } t \in \mathbb{R}\,.$$

Now again given $\epsilon > 0$ there is a $t'_\epsilon < \infty$ such that $f(t) \leqslant (\ell + \epsilon)e^{-(S+1/2)t}$ for $t \geqslant t'_\epsilon$. Inserting this and the previous bound on g in the equation for G' we obtain

$$G'(t) \geqslant -\beta_1(\ell-\epsilon)^3 e^{-4(S+1/2)t} - \operatorname{const} e^{-8(S+1/2)t} \qquad \text{for all } t \geqslant t'_\epsilon$$

and therefore by integration similarly as before

$$g(t) \leqslant \frac{\beta_1(\ell+\epsilon)^3}{4(S+1/2)} e^{-3(S+1/2)t} + \operatorname{const} e^{-7(S+1/2)t} \qquad \text{for all } t \geqslant t'_\epsilon\,.$$

This proves the claimed asymptotics for g as $t \to \infty$.

In order to obtain the asymptotics of f and g for $t \to -\infty$, we note that the pair $(g(-t), f(-t))$ solves (52). (Note that we have reversed the roles of f and g.) Therefore, applying the previous statement to this solution we obtain the claimed asymptotics for $t \to -\infty$.

Step 7. Uniqueness. We show that the non-trivial solution of (52) satisfying (53) is unique, up to translation and a sign change.

We give the argument only for $S + 1/2 > 0$, the case $S + 1/2 < 0$ being similar. We know from Step 6 that

$$\lim_{t\to\infty} \frac{g(t)}{f(t)} = 0 = \lim_{t\to-\infty} \frac{f(t)}{g(t)}\,.$$

Thus, by continuity there is a $t_0 \in \mathbb{R}$ such that $b := f(t_0) = g(t_0)$. Assumption (53) implies $\mathcal{E} = 0$ and therefore

$$\frac{\beta_1}{4} 2b^4 + \beta_2 b^4 - (S+1/2)b^2 = 0\,.$$

Thus, $b \in \{0, -a, +a\}$ with a defined in the theorem. Since the solution is non-trivial, we have $b \neq 0$. If (q, p) denotes the solution from Step 4, then by uniqueness of the solution of an initial value problem we have $(f(t), g(t)) = (q(t-t_0), p(t-t_0))$ if $b = a$ and $(f(t), g(t)) = (-q(t-t_0), -p(t-t_0))$ if $b = -a$. This proves the above uniqueness claim.

Step 8. Conclusion of the proof. We now prove all the statements of Theorem 1.5 translated to logarithmic variables.

Let (p, q) be the solution from Step 4 which we already know is global and satisfies (53). Therefore Step 6 describes the asymptotic behavior of this solution. The fact that $\tau q(t)p(t) > 0$ for all $t \in \mathbb{R}$ was already noted in Step 1 and the fact that $p(t) = \tau\, q(-t)$ for all $t \in \mathbb{R}$ follows from the fact that $(\tau\, q(-t), \tau\, p(-t))$ is a solution of (52) with the same values at $t = 0$ as $(p(t), q(t))$. This concludes the proof of part (1) of the theorem. Part (2) follows immediately from Step 7. □

References

[1] B. Ammann, *A variational problem in conformal spin geometry*, Habilitationsschift, Universität Hamburg, 2003.

[2] ———, *The smallest Dirac eigenvalue in a spin-conformal class and cmc immersions*, Comm. Anal. Geom., 17 (2009), pp. 429–479.

[3] B. Ammann, J.-F. Grosjean, E. Humbert, and B. Morel, *A spinorial analogue of Aubin's inequality*, Math. Z., 260 (2008), pp. 127–151.

[4] B. Ammann, E. Humbert, and B. Morel, *Mass endomorphism and spinorial Yamabe type problems on conformally flat manifolds*, Comm. Anal. Geom., 14 (2006), pp. 163–182.

[5] J. Arbunich and C. Sparber, *Rigorous derivation of nonlinear Dirac equations for wave propagation in honeycomb structures*, J. Math. Phys., 59 (2018), pp. 011509, 18.

[6] C. Bär, *Lower eigenvalue estimates for Dirac operators*, Math. Ann., 293 (1992), pp. 39–46.

[7] ———, *Zero sets of solutions to semilinear elliptic systems of first order*, Invent. Math., 138 (1999), pp. 183–202.

[8] T. Bartsch and T. Xu, *A spinorial analogue of the Brezis-Nirenberg theorem involving the critical Sobolev exponent*, ArXiv e-prints, (2018).

[9] A. Berthier and V. Georgescu, *On the point spectrum of Dirac operators*, J. Funct. Anal., 71 (1987), pp. 309–338.

[10] W. Borrelli, *Stationary solutions for the 2D critical Dirac equation with Kerr nonlinearity*, J. Differential Equations, 263 (2017), pp. 7941–7964.

[11] ———, *Weakly localized states for nonlinear Dirac equations*, Calc. Var. Partial Differential Equations, 57 (2018), p. 57:155.

[12] N. Boussaïd and A. Comech, *On spectral stability of the nonlinear Dirac equation*, J. Funct. Anal., 271 (2016), pp. 1462–1524.

[13] ———, *Nonrelativistic asymptotics of solitary waves in the Dirac equation with Soler-type nonlinearity*, SIAM J. Math. Anal., 49 (2017), pp. 2527–2572.

[14] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., 36 (1983), pp. 437–477.

[15] B. Cassano, *Sharp exponential localization for eigenfunctions of the Dirac Operator*, ArXiv e-prints, (2018).

[16] C. L. Fefferman and M. I. Weinstein, *Honeycomb lattice potentials and dirac points*, J. Amer. Math. Soc., 25 (2012), pp. 1169–1220.

[17] ———, *Waves in honeycomb structures*, Journées équations aux dérivées partielles, (2012).

[18] ———, *Wave packets in honeycomb structures and two-dimensional Dirac equations*, Comm. Math. Phys., 326 (2014), pp. 251–286.

[19] R. L. Frank and T. König, *Classification of positive singular solutions to a nonlinear biharmonic equation with critical exponent*, Anal. PDE, 12 (2019), pp. 1101–1113.

[20] T. Friedrich, *Dirac operators in Riemannian geometry*, vol. 25 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.

[21] L. Grafakos, *Classical Fourier analysis*, vol. 249 of Graduate Texts in Mathematics, Springer, New York, third ed., 2014.

[22] R. E. Greene and H. Wu, *Integrals of subharmonic functions on manifolds of nonnegative curvature*, Invent. Math., 27 (1974), pp. 265–298.

[23] N. Grosse, *On a conformal invariant of the Dirac operator on noncompact manifolds*, Ann. Global Anal. Geom., 30 (2006), pp. 407–416.

[24] ———, *Solutions of the equation of a spinorial Yamabe-type problem on manifolds of bounded geometry*, Comm. Partial Differential Equations, 37 (2012), pp. 58–76.

[25] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Comm. Math. Phys., 104 (1986), pp. 151–162.

[26] T. Isobe, *Nonlinear Dirac equations with critical nonlinearities on compact Spin manifolds*, J. Funct. Anal., 260 (2011), pp. 253–307.

[27] E. Jannelli and S. Solimini, *Concentration estimates for critical problems*, Ricerche Mat., 48 (1999), pp. 233–257. Papers in memory of Ennio De Giorgi (Italian).

[28] J. Jost, *Riemannian geometry and geometric analysis*, Universitext, Springer, Heidelberg, sixth ed., 2011.

[29] N. Korevaar, R. Mazzeo, F. Pacard, and R. Schoen, *Refined asymptotics for constant scalar curvature metrics with isolated singularities*, Invent. Math., 135 (1999), pp. 233–272.

[30] A. Loiudice, *L^p-weak regularity and asymptotic behavior of solutions for critical equations with singular potentials on Carnot groups*, NoDEA Nonlinear Differential Equations Appl., 17 (2010), pp. 575–589.

[31] A. Maalaoui, *Infinitely many solutions for the spinorial Yamabe problem on the round sphere*, NoDEA Nonlinear Differential Equations Appl., 23 (2016), pp. Art. 25, 14.

[32] R. O'Neil, *Convolution operators and $L(p, q)$ spaces*, Duke Math. J., 30 (1963), pp. 129–142.

[33] S. Raulot, *A Sobolev-like inequality for the Dirac operator*, J. Funct. Anal., 256 (2009), pp. 1588–1617.

[34] R. M. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, in Topics in calculus of variations (Montecatini Terme, 1987), vol. 1365 of Lecture Notes in Math., Springer, Berlin, 1989, pp. 120–154.

[35] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z., 187 (1984), pp. 511–517.

[36] G. Teschl, *Ordinary differential equations and dynamical systems*, vol. 140 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2012.

[37] F. Uguzzoni, *Asymptotic behavior of solutions of Schrödinger inequalities on unbounded domains of nilpotent Lie groups*, Rend. Sem. Mat. Univ. Padova, 102 (1999), pp. 51–65.

[38] J. Vétois, *Decay estimates and symmetry of finite energy solutions to elliptic systems in $\mathbb{R}^n$*, Indiana University Mathematics Journal. To appear, (2018).

[39] C. Wang, *A remark on nonlinear Dirac equations*, Proc. Amer. Math. Soc., 138 (2010), pp. 3753–3758.

(W. Borrelli) Université Paris-Dauphine, PSL Research University, CNRS, UMR 7534, CEREMADE, F-75016 Paris, France.

E-mail address: `borrelli@ceremade.dauphine.fr`

(R. L. Frank) Mathematisches Institut, Ludwig-Maximilans Universität München, Theresienstr. 39, 80333 München, Germany, and Mathematics 253-37, Caltech, Pasadena, CA 91125, USA

E-mail address: `rlfrank@caltech.edu`